Lecture note 17 (Oct 16, 2017)

- Midterm - Oct 30, 2017 (Monday) → in two weeks
- Quiz #3 - Oct 20, 2017 (Friday) → this Friday
  - Derivative (Definition) → lecture 15.
  - Derivative rules → lectures 16 & 17.
- Homework #5 will be posted (later today)
  - Due date: Oct 23, 2017 (Monday) → in one week

Last time, we saw the power Rule:

\[ \frac{d}{dx} (x^n) = (x^n)' = nx^{n-1} \]

Notation for "the derivative."

For example, let \( f(x) = x^{3/2} \) and \( g(x) = 5\sqrt{x^4} \), then

- \( f'(x) = (x^{3/2})' = \frac{3}{2} x^{3/2 - 1} = \frac{3}{2} x^{1/2} = \frac{3}{2} \sqrt{x} \)

and

- \( g'(x) = (5\sqrt{x^4})' = (x^{4/5})' = \frac{4}{5} x^{4/5 - 1} = \frac{4}{5} x^{-1/5} \)

\[ = \frac{4}{5} \frac{1}{x^{1/5}} = \frac{4}{5} \frac{4}{5\sqrt{x}} \]
Example 1: Can we use power rule to take the derivative of \( f(x) = 2^x \)? \( 2^x \) is an exponential function and not a polynomial.

(A) Yes \( \quad \) (B) No \( \quad \) (C) Don’t Know

What can we do? → Use the definition of the derivative. Let’s consider \( f(x) = 2^x \):

\[
\frac{f'(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \]

\[
= \cdots = 2^x \ln(2)
\]

So, \( (2^x)' = 2^x \ln(2) \)

In general,

\[\text{If } f(x) = b^x \text{, then } f'(x) = b^x \ln(b) \]

For example:

\[\text{if } f(x) = 10^x \text{, then } f'(x) = 10^x \ln(10)\]
In particular, for $f(x) = e^x$, we have
\[
\frac{d}{dx} f(x) = f'(x) = (e^x)' = e^x, \quad \ln(e) = e^x
\]

\[b = e\]

\[\text{slope} = \text{derivative} = e^x = e^2\]

at \[x = 2\]

\[e^2\]

"This is why $e^x$ is SO important"

Example 2: Find the derivative of $f(x) = xe^x + e^x$
\[
f'(x) = (xe^x + e^x)' = (xe^x)' + (e^x)'
\]
\[= e^x e^{-1} + e^x\]

\[\text{power rule} \quad \rightarrow \quad \text{formula}\]
Derivative = slope of graph of $f(x) = \sin x$:

- Positive slope
- Zero slope
- Negative slope

slope at $0$ is $1$.

Let's sketch the derivative.

We suspect that $\frac{d}{dx} (\sin x) = \cos x$.

This can be proved using the definition of the derivative.

$$\frac{d}{dx} (\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \ldots \text{ "lots of trig identities"}$$

$$= \cos x.$$

So,

$$\frac{d}{dx} (\sin x) = (\sin x)' = \cos x.$$
Following the same idea (we can sketch the derivative of cosine), one can see that

\[
\frac{d}{dx} (\cos x) = -\sin x
\]

Thus,

- \[
\frac{d}{dx} (\sin x) = (\sin x)' = \cos x
\]
- \[
\frac{d}{dx} (\cos x) = (\cos x)' = -\sin x
\]

⇒ Derivative Rules

1. \[
\frac{d}{dx} (cf(x)) = (cf(x))' = c f'(x)
\]
2. \[
\frac{d}{dx} (f(x) + g(x)) = (f(x) + g(x))' = f'(x) + g'(x)
\]
3. \[
\frac{d}{dx} (f(x) - g(x)) = (f(x) - g(x))' = f'(x) - g'(x)
\]

(5 \cos x)' = 5 (\cos x)' = 5 (\sin x) = -5 \sin x
Question: Is the following true?

\[(x \cdot \sin x)' = 1 \cdot \cos x ?\]

(A) Yes  (B) No  (C) Don't know, because we have not learned Product Rule yet.

The Product Rule:

\[(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)\]

Consider \(h(x) = (x+1)(x-2)\). What is the derivative of \(h(x)\)?

Method I: Simplify + Power Rule.

\[h(x) = x^2 - 2x + x - 2 = x^2 - x - 2. \checkmark\]

Power Rule:

\[h'(x) = (x^2 - x - 2)' = (x^2)' - (x)' - (2)' = 2x - 1 - 0 = 2x - 1. \checkmark\]
Method II: Product Rule:

\[ h(x) = \frac{(x+1)(x-2)}{f(x)g(x)} \]

\[ h'(n) = \left( \frac{(n+1)(n-2)}{f(n)g(n)} \right)' = \]

\[ = (n+1)'(n-2) + (n+1)(n-2)' \cdot g + f \cdot g' \]

\[ = 1 \cdot (n-2) + (n+1)(1) \]

- \( f'(n) = (n+1)' = n' + 1 = 1 + 0 = 1 \)
- \( g'(n) = (n-2)' = n' - 2 = 1 - 0 = 1 \)

\[ = n - 2 + n + 1 = 2n - 1 \checkmark \]

\[ (x \cdot \sin x)' = (x)' \sin x + x \cdot (\sin x)' \]

\[ = 1 \cdot \sin x + x \cos x \]

\[ = \sin x + x \cos x \]