8. Derivatives

Last class, we found the slope of the tangent lines to $f(x) = x^2$ at point $(2,4)$ and $(3,9)$:

We did this by taking the limit of secant lines. Indeed, we did one computation for each. Let's now find the slope of all tangent lines in one go.
Let’s find the slopes of all tangent lines in one go!

\[ f(x) = x^2 \]

\[ m_{\text{red line}} = m_{\text{sec}} = \frac{(x + h)^2 - x^2}{x + h - x} = \frac{(x + h)^2 - x^2}{h} \]

So, \[ m_{\text{blue line}} \]

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \]

Recall: \[ (x + h)^2 = (x + h)(x + h) = x^2 + 2xh + h^2 \]
So, we have

$$m_{\text{tan}} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

“This function tells us the slope of the tangent line to $x^2$ at any point”

For example:

at $x = 2$ slope is 4
at $x = 3$ slope is 6
at $x = 4$ slope is 8
at $x = -1$ slope is -2
If \( f(x) = x^2 \), then the slope of tangent line is the derivative of \( f(x) \) and we write:
\[
\frac{df}{dx} = f'(x) = 2x
\]
"Different notation for the derivative"

In general,

\[
\begin{align*}
    m_{\text{sec}} &= \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h} \\
    m_{\text{tan}} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} m_{\text{sec}}
\end{align*}
\]
Definition of the derivative

\[ f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} \]

Example 1: Find the derivative of \( f(x) = x^3 \).

\[ f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} \]
\[ = \lim_{{h \to 0}} \frac{(x+h)^3 - x^3}{h} \]
\[ = \cdots \]
\[ = 3x^2 \]