Lecture note 14 (Oct 6, 2017)

- Happy Thanksgiving
- Quiz #2 on Wed (Oct 11, 2017)
- HW #4 due on Wed (Oct 11, 2017)
- Quiz #2:
  - Lecture note 7, 8, 9, 10, 11 + First 3 pages of Lecture note 12
  - Exponential and logarithm function
    - limits
    - computing Limits

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Horizontal Asymptote (Continue…)

Consider functions:

\[ f(x) = e^x \]

\[ f(x) = e^{-x} \]

I. \( \lim_{x \to \infty} e^x = \infty \)

II. \( \lim_{x \to -\infty} e^x = 0 \)

III. \( \lim_{x \to \infty} e^{-x} = 0 \)

IV. \( \lim_{x \to -\infty} e^{-x} = \infty \)
Example 1: Find all Horizontal Asymptotes for

$$f(x) = \frac{-2}{e^x + 3}$$

In order to find H.A. of \(f(x)\), we compute the following limits:

(a) \(\lim_{x \to \infty} f(x) = L\) & (b) \(\lim_{x \to -\infty} f(x) = M\)

then \(y = L\) and \(y = M\) are H.A. for \(f(x)\).

(a) \(\lim_{x \to \infty} \frac{-2}{e^x + 3} \stackrel{(I)}{=} \frac{-2}{\infty + 3} \to 0\)

so, \(f(x)\) has H.A. at \(y = 0\).

(b) \(\lim_{x \to -\infty} \frac{-2}{e^x + 3} \stackrel{(II)}{=} \frac{-2}{0 + 3} = \frac{-2}{3}\)

so, \(f(x)\) has H.A. at \(y = -\frac{2}{3}\)
Slope of the Tangent lines:

we want to find the slope of the tangent line:

- speed (instantaneous velocity)
- rate of change

Finding the slope of the tangent line is hard, since we only have one point. Easier is finding the slope of ... secant line.
we start by finding the slope of secant line through $(2,4)$ and $(3,9)$:

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 4}{3 - 2} = \frac{5}{1} = 5 \rightarrow \text{approximation of the tangent line}$$

"the slope of the secant line"

A better approximation would be to use $(2.5, 6.25)$

$$m_{sec} = \frac{6.25 - 4}{2.5 - 2} = \frac{2.25}{0.5} = 4.5 \rightarrow \text{"better"}$$

Even better is $(2.1, 4.41)$

$$m_{sec} = \frac{4.41 - 4}{2.1 - 2} = \frac{0.41}{0.1} = 4.1 \rightarrow \text{"even better"}$$

To find the slope exactly,

we take the limit.
\[ m_{\text{sec}} = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{(2+h)^2 - 4}{h} \]

As \( h \) gets smaller, our \( m_{\text{sec}} \) gets closer to \( m_{\text{tan}} \) (the slope of the tangent line).

Consider the limit:

\[ \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} m_{\text{sec}} = m_{\text{tan}} \]

Let's compute this limit → substitution will not work → \( \frac{0}{0} \).
\[ m_{\text{tan}} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} \frac{h^2 + 4h + 4 - 4}{h} \]

\[ = \lim_{h \to 0} \frac{h^2 + 4h}{h} \]

\[ = \lim_{h \to 0} (h + 4) \]

\[ = 4 \]

"the slope of the tangent line at \( x = 2 \) for \( f'(m_1) = x^2 \)."
Example 2: Find the slope of the tangent line to \( f(x) = x^2 \) at the point \((3, 9)\).

\[
m_{\text{tan}} = \lim_{h \to 0} \frac{(3+h)^2 - 9}{(3+h) - 3} = \lim_{h \to 0} \frac{(3+h)^2 - 9}{h}
\]

\[
= \lim_{h \to 0} \frac{(h^2 + 6h + 9) - 9}{h}
\]

\[
= \lim_{h \to 0} \frac{h^2 + 6h}{h}
\]

\[
= \lim_{h \to 0} \frac{h(h+6)}{h}
\]

\[
= \lim_{h \to 0} h + 6 = 0 + 6 = 6 \quad \checkmark
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