

Homework #7 - Solution

Question 1: It's Question 2 part (b) in Final Exam 2015. Its solution is posted.

Question 2:

(a) Let \( u = \sin x \), then \( du = \cos x \, dx \). So,

\[
\int \frac{\cos x}{\sin x} \, dx = \int \frac{\cos x}{u} \frac{du}{\cos x} = \int \frac{du}{u} = \ln u + C \quad \to \sin x
\]

\[= \ln (\sin x) + C.\]

(b) Let \( u = 2 + e^x \), then \( du = e^x \, dx \). So,

\[
\int e^x \sin (2+e^x) \, dx = \int e^x \sin (u) \frac{du}{e^x} = \int \sin u \, du
\]

\[= -\cos u + C = -\cos (2+e^x) + C \quad \to 2+e^x
\]

(c) Let \( u = x+1 \), then \( du = dx \). So,

\[
\int (x-2) \sqrt{x+1} \, dx = \int (x-2) \sqrt{u} \, du
\]
So, we need to find \( x - 2 \) in the integral in terms of \( u \). For that, we have
\[
u = x + 1 \implies x = u - 1 \implies x - 2 = u - 3.
\]
In other words,
\[
\int (x - 2) \sqrt{x + 1} \, \text{d}m = \int (u - 2) \sqrt{u} \, \text{d}u = \int ((u - 1) - 1) \sqrt{u} \, \text{d}u = \int (u - 3) \sqrt{u} \, \text{d}u = \int (u^{3/2} - 3u^{1/2}) \, \text{d}u
\]
\[
= \frac{2}{5} u^{5/2} - 3 \frac{2}{3} u^{3/2} + C = \frac{2}{5} u^{5/2} - 2 u^{3/2} + C
\]
\[
= \frac{2}{5} (u + 1)^{5/2} - 2 (u + 1)^{3/2} + C.
\]

(d) Let \( u = \cos x \), then \( \text{d}u = -\sin x \, \text{d}x \). So,
\[
\int \frac{\sin x}{\cos x} (\cos x + 1) \, \text{d}m = \int \frac{\sin x (u + 1)}{u} \, \text{d}u = -\int \frac{u + 1}{u} \, \text{d}u
\]
\[
= -\int \left( 1 + \frac{1}{u} \right) \, \text{d}u = -\int (1 + \frac{1}{u}) \, \text{d}u = -(u + \ln u) + C
\]
\[
= -u - \ln u + C = \boxed{-\cos x - \ln(\cos x) + C}
\]
(b) Similar to "Example 1" in lecture note 31. The final answer is $\sin x - x \cos x + C$

(c) Similar to "Example 2" in lecture note 33. The final answer is $2 \sin x + 2x \cos x - 2 \sin x + C$

(d) Solution is in "Example 3" in lecture note 33.

**Question 4:** We use substitution + IBP

Let $u = \sin x$, then $du = \cos x \, dx$. So,

$$\int \cos x \ln(\sin x) \, dx = \int \cos x \ln u \frac{du}{\cos x} = \int \ln u \, du.$$ 

For the last integral, we use integration by parts. We have already seen in Question 3 part (d) that

$$\int u \ln u \, du = u \ln u - u + C.$$ 

We plug $u = \sin x$ back to get that

$$\int \cos x \ln(\sin x) \, dx = u \ln u - u + C = \sin x \ln(\sin x) - \sin x + C.$$