Questions:

1. Consider the function

\[ f(x) = -\frac{1}{2}x + 5 \]

(a) Approximate the area under the curve on the interval \([0, 4]\) using Riemann Sums. Use left endpoints and two bars \((n = 2)\).

(b) Now approximate the same area using four bars \((n = 4)\), again with left endpoints.

(c) Compute the exact area, either by integrating or by drawing a picture and using area formulas. Which approximation is better? Are your approximations over or under estimates? Explain why you would expect this at the start of the problem, perhaps in reference to your picture.

Solution: (a) We first approximate the area using two bars. See the figure below. We can represent this sum in the following way

\[ \sum_{i=0}^{1} f(x_i) \Delta x \]

where \(\Delta x\) is the width of the bars and the \(x_i\) are the left endpoints. We have

\[ \Delta x = \frac{b - a}{n} = \frac{4 - 0}{2} = 2 \]
and $x_0 = 0$ and $x_1 = 2$. In this way we approximate the integral as follows

$$\sum_{i=0}^{1} f(x_i) \Delta x = f(x_0) \Delta x + f(x_1) \Delta x$$

$$= 5 \cdot 2 + 4 \cdot 2$$

$$= 18.$$ 

So our approximation of the area is 18.

(b) Now we approximate the area using four bars. Observe again the figure. We can represent this sum in the following way

$$\sum_{i=0}^{3} f(x_i) \Delta x$$

where $\Delta x = (4 - 0)/4 = 1$ and $x_i = i$. We compute

$$\sum_{i=0}^{3} f(x_i) \Delta x = f(x_0) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x$$

$$= 5 \cdot 1 + 4.5 \cdot 1 + 4 \cdot 1 + 3.5 \cdot 1$$

$$= 17$$

and so find 17 as our approximation.

(c) Let us compute the exact area. This can be done by splitting the area under the curve into a triangle and a rectangle. In this way

$$\int_{0}^{4} \left( -\frac{1}{2} x + 5 \right) dx = 4 \cdot 3 + \frac{2 \cdot 4}{2} = 12 + 4 = 16.$$ 

We can also compute the integral directly using the Fundamental Theorem of Calculus

$$\int_{0}^{4} \left( -\frac{1}{2} x + 5 \right) dx = -\frac{1}{4} x^2 + 5x \bigg|_{x=0}^{x=4} = -\frac{1}{4} 4^2 + 5 \cdot 4 = 16.$$
In light of the above we achieve the exact area, 16. We notice that the second approximation is better, having used more bars our computed area is closer to the exact area. Both estimates are overestimates. This is to be expected after observing each graph since we see our bars are capturing additional area beyond the desired region.

2. Consider the function

\[ g(x) = \begin{cases} 
2(x - 1) + 6, & x \leq 1 \\
-x + 7, & x > 1 
\end{cases} \]

(a) Sketch the graph of \( g(x) \).

(b) Compute 

\[ \int_{-1}^{3} g(x) \, dx. \]

**Solution:** First, we sketch a graph of \( g(x) \).

We seek the area under this curve between \( x = -1 \) and \( x = 3 \). We will split the integral into two parts, these being the area under the curve between \( x = -1 \) and \( x = 1 \) and then the area under the curve between \( x = 1 \) and \( x = 3 \). That is we compute

\[ \int_{-1}^{3} g(x) \, dx = \int_{-1}^{1} g(x) \, dx + \int_{1}^{3} g(x) \, dx. \]

We know that between \( x = -1 \) and \( x = 1 \) the function \( g(x) \) takes the first branch and so we
substitute this line into the first integral. We similarly compute the second integral. Observe

\[
\int_{-1}^{3} g(x) \, dx = \int_{-1}^{1} g(x) \, dx + \int_{1}^{3} g(x) \, dx
\]

\[
= \int_{-1}^{1} (2(x - 1) + 6) \, dx + \int_{1}^{3} (-x + 7) \, dx
\]

\[
= \int_{-1}^{1} (2x + 4) \, dx + \int_{1}^{3} (-x + 7) \, dx
\]

\[
= \left[ x^2 + 4x \right]_{-1}^{1} + \left[ -\frac{x^2}{2} + 7x \right]_{1}^{3}
\]

\[
= 1 + 4 - (1 - 4) - \frac{9}{2} + 7 \cdot 3 - \left( -\frac{1}{2} + 7 \right)
\]

\[
= 8 - 4 + 14
\]

\[
= 18.
\]

Alternatively we could split the total area into two rectangles and two triangles to achieve the same result. That is

\[
\int_{-1}^{3} g(x) \, dx = 2 \cdot 2 + 2 \cdot 4 + 2 \cdot 4 + 2 \cdot \frac{2}{2}
\]

\[
= 4 + 8
\]

\[
= 18.
\]
3. If we know that
   \( \int_{-1}^{3} f(x) \, dx = 6 \)
   \( \int_{-1}^{3} g(x) \, dx = -3 \)
then compute
(a) \( \int_{-1}^{3} (2f(x) - g(x)) \, dx \)
(b) \( \int_{-1}^{3} (4f(x) + 5g(x)) \, dx \)

**Solution:** (a) We split the integral in order to use our known information. Observe
\[
\int_{-1}^{3} (2f(x) - g(x)) \, dx = \left[ 2f(x) \right]_{-1}^{3} - \left[ g(x) \right]_{-1}^{3}
\]
\[
= 2(6) - (-3)
\]
\[
= 12 + 3
\]
\[
= 15.
\]
(b) We apply the same manipulation again to see
\[
\int_{-1}^{3} (4f(x) + 5g(x)) \, dx = \left[ 4f(x) \right]_{-1}^{3} + 5 \left[ g(x) \right]_{-1}^{3}
\]
\[
= 4(6) + 5(-3)
\]
\[
= 24 - 15
\]
\[
= 9.
\]

4. Compute the following definite integral
(a) \( \int_{0}^{\pi} (\cos x + x^3 + 4) \, dx \) and (b) \( \int_{1}^{3} \left( \frac{1}{\sqrt{x^3}} - 3e^x + \frac{5}{x} \right) \, dx \)

**Solution:**
(a) We compute using the Fundamental Theorem of Calculus
\[
\int_{0}^{\pi} (\cos x + x^3 + 4) \, dx = \left[ \sin x + \frac{1}{4}x^4 + 4x \right]_{0}^{\pi}
\]
\[
= \sin(\pi) + \frac{\pi^4}{4} + 4\pi - \left( \sin(0) + \frac{0^4}{4} + 4(0) \right)
\]
\[
= \frac{\pi^4}{4} + 4\pi.
\]
(b) We compute using the Fundamental Theorem of Calculus

\[
\int_1^3 \left( \frac{1}{\sqrt[3]{x^3}} - 3e^x + \frac{5}{x} \right) dx = \int_1^3 \left( x^{-\frac{3}{2}} - 3e^x + \frac{5}{x} \right) dx = \frac{1}{-\frac{3}{2} + 1} x^{-\frac{3}{2}+1} - 3e^x + 5 \ln |x| \bigg|_1^3
\]

\[
= -2x^{-\frac{1}{2}} - 3e^x + 5 \ln |x| \bigg|_1^3 = \frac{-2}{\sqrt{x}} - 3e^x + 5 \ln |x| \bigg|_1^3
\]

\[
= \left( \frac{-2}{\sqrt{3}} - 3e^3 + 5 \ln |3| \right) - \left( \frac{-2}{\sqrt{1}} - 3e^1 + 5 \ln |1| \right)
\]

\[
= \frac{-2}{\sqrt{3}} + 2 - 3e^3 + 3e + 5 \ln 5.
\]