Review [Parametric description of a circle/an ellipse]

<table>
<thead>
<tr>
<th></th>
<th>A circle of radius $a$ centered at $(0,0)$</th>
<th>An ellipse centered at $(0,0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General Form</strong></td>
<td>$x^2 + y^2 = a^2$</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$</td>
</tr>
<tr>
<td><strong>Parametric Equation</strong></td>
<td>$x = a \cos \theta$</td>
<td>$x = a \cos \theta$</td>
</tr>
<tr>
<td></td>
<td>$y = a \sin \theta$, for $0 \leq \theta \leq 2\pi$</td>
<td>$y = b \sin \theta$, for $0 \leq \theta \leq 2\pi$</td>
</tr>
</tbody>
</table>

**Why?!!**

```
\[ \begin{align*}
\cos \theta &= \frac{x}{a} \\
\sin \theta &= \frac{y}{a}
\end{align*} \implies \begin{align*}
x &= a \cos \theta \\
y &= a \sin \theta,
\end{align*} \text{ for } 0 \leq \theta \leq 2\pi
```

θ is changing from 0 to $2\pi$.

Check: (ellipse)

$x = a \cos \theta$ and $y = b \sin \theta$.

Plug into $\frac{x^2}{a^2} + \frac{y^2}{b^2}$:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(a \cos \theta)^2}{a^2} + \frac{(b \sin \theta)^2}{b^2} = \frac{a^2 \cos^2 \theta}{a^2} + \frac{b^2 \sin^2 \theta}{b^2} = \cos^2 \theta + \sin^2 \theta = 1$.

So, by choosing $x = a \cos \theta$ and $y = b \sin \theta$, we have

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

<table>
<thead>
<tr>
<th></th>
<th>A circle of radius $a$ centered at the point $(x_0, y_0)$</th>
<th>An ellipse centered at the point $(x_0, y_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General Form</strong></td>
<td>$(x-x_0)^2 + (y-y_0)^2 = a^2$</td>
<td>$(x-x_0)^2 + \frac{(y-y_0)^2}{b^2} = 1$</td>
</tr>
<tr>
<td><strong>Parametric Equation</strong></td>
<td>$x = a \cos \theta + x_0$</td>
<td>$x = a \cos \theta + x_0$</td>
</tr>
<tr>
<td></td>
<td>$y = a \sin \theta + y_0$, for $0 \leq \theta \leq 2\pi$</td>
<td>$y = b \sin \theta + y_0$, for $0 \leq \theta \leq 2\pi$</td>
</tr>
</tbody>
</table>
Let’s go back and take a look at the last example that we solved in the lecture 8. Our objective function was \( f(x,y) = 5x - 3y \) and the constraint \( g(x,y) = x^2 + y^2 - 136 = 0 \). So, we got

\[
\begin{align*}
\nabla f(x,y) &= \lambda \nabla g(x,y) \\
g(x,y) &= 0
\end{align*}
\]

\[\Rightarrow (**) \]

\[
\begin{align*}
\frac{f_x}{g_x} &= \lambda \\
\frac{f_y}{g_y} &= \lambda \\
g(x,y) &= 0
\end{align*}
\]

\[\Rightarrow \begin{cases} 
5 = \lambda (2x) \\
-3 = \lambda (2y) \\
x^2 + y^2 - 136 = 0
\end{cases}
\]

Now, to find \( x \) and \( y \) from (**) , we can do one of the following:

1. [we already did this one] Find \( x \) and \( y \) in terms of \( \lambda \), then plug them into the constraint \( g(x,y) = 0 \). So, we’ll find \( \lambda \), and then \( x \) and \( y \)!

\[
\begin{align*}
x &= \frac{5}{2\lambda} \\
y &= \frac{-3}{2\lambda}
\end{align*}
\]

plug into \( 25 \frac{4}{4\lambda^2} + 9 \frac{-136}{4\lambda^2} = 0 \) \( \Rightarrow \) solve for \( \lambda \) \( \Rightarrow \lambda = \pm \frac{1}{4} \)

\[\Rightarrow \begin{cases} 
x &= 10 \\
y &= -6
\end{cases}
\]

if \( \lambda = \frac{1}{4} \): \( \begin{cases} 
x &= 10 \\
y &= -6
\end{cases} \)

if \( \lambda = \frac{1}{4} \): \( \begin{cases} 
x &= 10 \\
y &= 6
\end{cases} \)

2. Find \( \lambda \) in terms of \( x \) and \( y \) in (I) and (II), separately, set them equal. It gives us the relation between \( x \) and \( y \). Then, plug it into \( g(x,y) = 0 \) to find \( x \) and \( y \)!

\[
\begin{align*}
\text{(I)} \Rightarrow \lambda &= \frac{5}{2x} \\
\text{(II)} \Rightarrow \lambda &= \frac{-3}{2y}
\end{align*}
\]

plug into \( 25 \frac{5}{2x} = \lambda = \frac{-3}{2y} \) \( \Rightarrow \)

plug into \( g(x,y) = 0 \) \( \Rightarrow \)

\[\frac{2}{-3} \frac{x}{2y} = 136 \]

\[\Rightarrow x = \pm 10 \Rightarrow \begin{cases} 
x &= 10 \Rightarrow y = -6 \end{cases} \text{ and } \begin{cases} 
x &= -10 \Rightarrow y = 6 \end{cases}
\]

3. Divide (I) by (II) to find the relation between \( x \) and \( y \). Then, plug it into \( g(x,y) = 0 \) to find \( x \) and \( y \)!

\[
\text{(I)} \Rightarrow \frac{5}{-3} = \frac{x}{2y} \Rightarrow \frac{y}{2x} = \frac{-3}{5} \Rightarrow \begin{cases} 
x &= 10, y = -6 \\
x &= 10, y = 6
\end{cases}
\]

Notes: if \( \lambda = 0 \), \( 5 = 0(2x) = 0 \) and \( -3 = 0(2y) = 0 \), which is impossible. So, \( \lambda \) cannot be 0. The same reason \( \Rightarrow x \neq 0, y \neq 0 \).
Example 1: Use Lagrange multipliers to find the maximum and minimum values of \( f(x,y) \) subject to the given constraint.

- \( f(x,y) = x^2 + y^2 \) subject to \( 2x^2 + 3xy + 2y^2 = 7 \).

The objective \( f(x,y) = x^2 + y^2 \).

The constraint \( g(x,y) = 2x^2 + 3xy + 2y^2 - 7 = 0 \).

\[ \nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle \]
\[ \nabla g = \langle g_x, g_y \rangle = \langle 4x + 3y, 3x + 4y \rangle \]

\[ \begin{align*}
  f_x &= 2x = \lambda (4x + 3y) = \lambda g_x \quad \text{(I)} \\
  f_y &= 2y = \lambda (3x + 4y) = \lambda g_y \quad \text{(II)}
\end{align*} \]

Now, if \( \lambda = 0 \), we have \( 2x = 0 \) and \( 2y = 0 \), which means \( x = y = 0 \). But, you can see \( x = y = 0 \) does not satisfy the constraint \( 2x^2 + 3xy + 2y^2 = 7 = 0 \). So, \( \lambda \neq 0 \).

If \( \lambda \neq 0 \), we can divide (I) by (II) to get

\[ \frac{2x}{2y} = \frac{\lambda (4x + 3y)}{\lambda (3x + 4y)} \quad \implies \quad \frac{x}{y} = \frac{4x + 3y}{3x + 4y} \]

\[ \implies 3x^2 + 4yx = 4xy + 3y^2 \quad \implies \quad y^2 = x^2 \quad \implies \quad y = \pm x \]
If $y = x$:

\[
\begin{align*}
\begin{cases}
  y = x \\
g(x, y) = 0
\end{cases} \quad \Rightarrow \quad g(x, x) = 0 \quad \Rightarrow \quad 2x^2 + 3x(x) + 2(x)^2 = 7 \\
\Rightarrow \quad 7x^2 = 7 \quad \Rightarrow \quad x = \pm 1
\end{align*}
\]

\[
\begin{cases}
  x = 1 \quad \Rightarrow \quad y = x = 1 \\
x = -1 \quad \Rightarrow \quad y = x = -1
\end{cases}
\]

\[
\begin{align*}
\begin{pmatrix}
(1, 1) \\
(-1, -1)
\end{pmatrix}
\end{align*}
\]

If $y = -x$:

\[
\begin{align*}
\begin{cases}
  y = -x \\
g(x, y) = 0
\end{cases} \quad \Rightarrow \quad g(x, -x) = 0 \quad \Rightarrow \quad 2x^2 + 3x(-x) + 2(-x)^2 = 7 \\
\Rightarrow \quad x^2 = 7 \quad \Rightarrow \quad x = \pm \sqrt{7}
\end{align*}
\]

\[
\begin{cases}
  x = +\sqrt{7} \quad \Rightarrow \quad y = -x = -\sqrt{7} \\
x = -\sqrt{7} \quad \Rightarrow \quad y = -x = \sqrt{7}
\end{cases}
\]

\[
\begin{align*}
\begin{pmatrix}
(\sqrt{7}, -\sqrt{7}) \\
(-\sqrt{7}, \sqrt{7})
\end{pmatrix}
\end{align*}
\]

So,

\[
\begin{align*}
f(1, 1) = f(1(-1)) &= 2 \\
f(\sqrt{7}, -\sqrt{7}) = f(-\sqrt{7}, \sqrt{7}) &= 14
\end{align*}
\]

\[
\begin{align*}
\text{Min} \\
\text{Max}
\end{align*}
\]
To find the relation between $x$ and $y$, you can also multiply (I) by $y$ and (II) by $x$, and subtract to get [example 1]:

$$y \cdot (\text{I}) \Rightarrow 2xy = \lambda (4xy + 3y^2)$$
$$x \cdot (\text{II}) \Rightarrow 2xy = \lambda (3x^2 + 4xy)$$

Subtracting these equations gives:

$$2xy - 2xy = \lambda (4xy + 3y^2) - \lambda (3x^2 + 4xy) = \lambda (3x^2 - 3y^2) = 0$$

So, we get $\lambda (3x^2 - 3y^2) = 0 \Rightarrow \lambda = 0$ or $y^2 = x^2$. As we saw before, $\lambda$ cannot be zero, so $y = \pm x$. The rest would be the same.

**Example 2.** A manufacturer's production is modeled by the Cobb-Douglas function

$$u(x, y) = 100x^{\frac{4}{5}}y^{\frac{1}{5}}$$

where $x$ represents the units of labor and $y$ represents the units of capital. Each labor unit costs $200$ and each capital unit costs $250$. The total expenses for labor and capital cannot exceed $50,000$. Find the maximum production level.

The objective function is $f(x, y) = 100x^{\frac{4}{5}}y^{\frac{1}{5}}$.

The constraint comes from the sentence "The total expenses for labor and capital cannot exceed $50,000". So,

$$g(x, y) = 200x + 250y - 50,000 = 0 \quad \text{(constraint)}$$

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 80x^{\frac{1}{5}}y^{\frac{4}{5}}, 20x^{\frac{4}{5}}y^{\frac{1}{5}} \rangle$$

$$\nabla g(x, y) = \langle 200, 250 \rangle$$
\( \nabla f = \lambda \nabla g \) implies that 

\[ \begin{align*}
1. & \quad 80x - \frac{3}{5}y^\frac{4}{5} = 200\lambda \\
2. & \quad 20x^\frac{4}{5}y^{-\frac{2}{5}} = 250\lambda
\end{align*} \]

we have 

\[ \lambda = \begin{cases} 
\frac{80x^\frac{4}{5}y^{-\frac{2}{5}}}{200} & \text{From (i)} \\
\frac{20x^\frac{4}{5}y^{-\frac{2}{5}}}{250} & \text{From (ii)}
\end{cases} \]

\[ \begin{align*}
\frac{80}{200} \cdot y^\frac{1}{5} = \lambda = \frac{20}{250} \cdot x^\frac{4}{5}, \quad \text{so,}
80 \cdot 250 \cdot y = 20 \cdot 200 \cdot x
\end{align*} \]

\[ 5y = x \]

Now, plug \( 5y = x \) into \( g(x,y) = 0 \) to get 

\[ 200(5y) + 250y = 50,000. \]

So, \( y = 40 \), and \( x = 4(40) = 200 \). The point is \((200,40)\).

\( f(200,40) = 100 \cdot (200)^\frac{4}{5} \cdot (40)^\frac{1}{5} \).

Example 3. Use Lagrange multipliers to find the maximum and minimum values of \( xy^2 \) on the ellipse \( 2x^2 + y^2 = 1 \).

the objective function \( f(x,y) = xy^2 \).

the constraint \( g(x,y) = 2x^2 + y^2 - 1 = 0 \).

Let's find the gradient of \( f(x,y) \) and \( g(x,y) \).

\( \nabla f(x,y) = \langle f_x, f_y \rangle = \langle y^2, 2xy \rangle \)

\( \nabla g(x,y) = \langle g_x, g_y \rangle = \langle 4x, 2y \rangle \).
So,
\[ \begin{align*}
\n\n\n\begin{cases}
\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\na = \frac{y^2}{4x} \text{ and } \lambda = \frac{2xy}{2x} = x \Rightarrow \frac{y^2}{4x} = \lambda = x \Rightarrow y^2 = 4x^2. \quad \text{Now, by plugging } y^2 = 4x^2 \text{ into the constant } 2x^2 + y^2 - 1 = 0, \text{ we get}

\text{if } y = 2x, \Rightarrow y = 2x \Rightarrow g(x, y) = 0 \Rightarrow g(x, 2x) = 0 \Rightarrow 2x^2 + (2x)^2 - 1 = 0

\Rightarrow 6x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{6}} \Rightarrow x = \pm \frac{1}{\sqrt{6}}

\begin{align*}
\{ x = \frac{1}{\sqrt{6}} \Rightarrow y = 2x = 2\sqrt{\frac{1}{6}} \} & \Rightarrow \left( \frac{\sqrt{1}}{\sqrt{6}}, 2\sqrt{\frac{1}{6}} \right) \\
\{ x = -\frac{1}{\sqrt{6}} \Rightarrow y = 2x = -2\sqrt{\frac{1}{6}} \} & \Rightarrow \left( -\frac{1}{\sqrt{6}}, -2\sqrt{\frac{1}{6}} \right)
\end{align*}
If \( y = -2x \):

\[
\begin{align*}
g(x,y) &= 2x^2 + y^2 - 1 = 0 \\
y &= -2x
\end{align*}
\]

\[\Rightarrow g(x,-2x) = 0 \Rightarrow 2x^2 + (-2x)^2 - 1 = 0 \Rightarrow 6x^2 = 1 \Rightarrow x = \pm \sqrt{\frac{1}{6}}.
\]

\[
\begin{align*}
\{ \frac{x}{\sqrt{\frac{1}{6}}} & \Rightarrow y = -2x = -2\sqrt{\frac{1}{6}} \} \Rightarrow (\sqrt{\frac{1}{6}}, -2\sqrt{\frac{1}{6}}) \\
\{ \frac{x}{-\sqrt{\frac{1}{6}}} & \Rightarrow y = -2x = -2(-\sqrt{\frac{1}{6}}) \} \Rightarrow (-\sqrt{\frac{1}{6}}, 2\sqrt{\frac{1}{6}})
\end{align*}
\]

So, we have the four points:
\( (\sqrt{\frac{1}{6}}, 2\sqrt{\frac{1}{6}}), (-\sqrt{\frac{1}{6}}, -2\sqrt{\frac{1}{6}}), (\sqrt{\frac{1}{6}}, -2\sqrt{\frac{1}{6}}), (-\sqrt{\frac{1}{6}}, 2\sqrt{\frac{1}{6}}) \).

The values of \( f(x,y) \) at these points:

\[
\begin{align*}
f(\sqrt{\frac{1}{6}}, 2\sqrt{\frac{1}{6}}) &= f(\sqrt{\frac{1}{6}}, -2\sqrt{\frac{1}{6}}) = \frac{4\sqrt{\frac{1}{6}}}{6} = \frac{2\sqrt{\frac{1}{6}}}{3} \\
f(-\sqrt{\frac{1}{6}}, 2\sqrt{\frac{1}{6}}) &= f(-\sqrt{\frac{1}{6}}, -2\sqrt{\frac{1}{6}}) = \frac{-4\sqrt{\frac{1}{6}}}{6} = -\frac{2\sqrt{\frac{1}{6}}}{3}
\end{align*}
\]

So,
\[
\begin{align*}
\frac{2}{3} \sqrt{\frac{1}{6}} & \text{ max} \\
-\frac{2}{3} \sqrt{\frac{1}{6}} & \text{ min}
\end{align*}
\]