8. Absolute and Conditional Convergence

So far we have only built strategies to determine convergence of series with positive terms (Integral test, comparison, limit comparison tests). All require you to only look at positive sums. Now, if you allow negative terms appear in your series, things get a bit different. Here, we’ll consider $\sum a_n$ as a series of positive terms/or an alternating series/or even a more general infinite series.

Let’s take a look at the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

\(\Rightarrow\) converges.

Using methods that you are not responsible for (look up alternating series test if interested), one can show that the above series **Converges**.
on the other hand, we have
\[ \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}, \]
and we know that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

In summary, by this example we can see:
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.} \]

In this case, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) is called "Converges Conditionally."

Now, we consider another example:
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \rightarrow \text{"Converges"} \]
Again, using the same method, we can show that the above series Converges. (Alternating Test). Also, we have
\[ \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}, \] and by p-test (p=2), we get that this series converges. Thus, \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) both converge. In this case, the series is called "Converges Absolutely."
Let’s summarize it as a definition:

**Definition (Absolute and Conditional Convergence)**

- If $\sum |a_n|$ (and also $\sum a_n$) converges, then $\sum a_n$ converges absolutely.

- If $\sum |a_n|$ diverges and $\sum a_n$ converges, then $\sum a_n$ converges conditionally.

Finally, we introduce the following theorem which is very useful later.

**Theorem (Absolute Convergence Implies Convergence)**

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Equivalently,

If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.
**Power Series**

So far, we have been interested in adding up numbers. But this is calculus, we need functions to play with. Let us now add functions, and define a function based on series. Indeed, we make a seemingly small, but significant, change by considering infinite series whose terms include powers of a variable. With this change, an infinite series becomes a power series.

- **Why Power Series?**
  - One of the most fundamental ideas in all of calculus is that functions can be represented by power series!

- **What Is a Power Series?**
  - A power series is an infinite series of the form
    \[
    \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + c_{n+1} x^{n+1} + \cdots
    \]
    or, more generally,
    \[
    \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + c_{n+1} (x-a)^{n+1} + \cdots
    \]
  - This type of series is called power series because it consists of powers of \( x \) or \( (x-a) \).
1.1 Convergence of Power Series

**Definition** (power series)

A Power Series centered at "a" is a function of the form

\[ \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \]

- The \( c_n \)'s are called the coefficients of the series.
- "a" is called the center of the series.
- "R" is called the radius of convergence, and is the distance from the center \( (a) \) to the boundary of the interval of convergence.

**Theorem** (Convergence of power series)

For a given power series \( \sum_{n=0}^{\infty} c_n (x-a)^n \). There are 3 possibilities:

1. There is a \( R > 0 \) such that \( |x-a| < R \rightarrow \) the series converges absolutely.
2. \( |x-a| > R \rightarrow \) the series diverges.
3. The series converges only at \( x=a \) \( \rightarrow \) the radius \( R = 0 \).
4. The series converges absolutely for all \( x \in \mathbb{R} \) \( \rightarrow \) the radius \( R = \infty \).
1. Properties of Power Series

- A good way to become familiar with power series:
  - Consider the following geometric series

\[
\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots = \frac{1}{1-r}, \quad \text{provided } -1 < r < 1
\]

Now, we do a small change and replace the real number "r" with the variable "x". Thus, we get

\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad -1 < x < 1 \quad \text{or} \quad |x| < 1
\]

This series is a power series (important) and it's a representation of the function \( \frac{1}{1-x} \) that is valid on the interval \(-1 < x < 1 \) (or \( |x| < 1 \)) !!!

\[
1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

interval \(-1 < x < 1 \) (or \( |x| < 1 \))
4. How can we find the interval and radius of convergence for a power series, \( \sum_{n=0}^{\infty} c_n(x-a)^n \)?

We test the series \( \sum_{n=0}^{\infty} c_n(x-a)^n \) for absolute convergence using the Ratio Test. We introduce 2 ways:

**Way 1**

Step 1: Find \( \lim_{n \to \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = r \)

Step 2: Let \( r < 1 \) condition in the ratio test for convergence.

Step 3: Solve \( r = \lim_{n \to \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| < 1 \), and find the interval for \( x \). Radius \( R = r \) distance of \( a \) to the boundary of the interval.

**Way 2**

Step 1: Find \( c_n \), and evaluate \( L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| \).

Step 2: \( R = \frac{1}{L} \) is the radius of convergence.

Step 3: Solve \( |x-a| < R \) to find the interval of convergence, that is \( -R + a < x < R + a \).
Example 1 (Interval and radius of convergence)

\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \longrightarrow \begin{cases} c_k = \frac{1}{k!} \\ a_0 = 0 \end{cases} \]

Method 1:

**Step 1:** Find 
\[ L = \lim_{n \to \infty} \frac{c_{k+1}}{c_k} = \lim_{n \to \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k \to \infty} \frac{k!}{(k+1)!} = 0 \]

**Step 2:** 
\[ R = \frac{1}{L} = \frac{1}{0} = \infty \rightarrow \text{Radius of Convergence} = R = \infty \]

In other words, the series converges for all \( x \in \mathbb{R} \).

**Step 3:** Interval of convergence:

\[ \left\{ \begin{array}{l} |x| < R \\ \text{and} \\ R = \infty \end{array} \right\} \rightarrow |x| < \infty \rightarrow -\infty < x < \infty \]

It shows the interval is all \( x \in \mathbb{R} \).

It means this series converges for all \( x \in \mathbb{R} \).
Method 2: Apply the ratio test to $a_k = \frac{x^k}{k!}$:

Find $r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k (k+1)!} \cdot \frac{k!}{k} \right| = \lim_{k \to \infty} \frac{|x|}{k+1} = \lim_{k \to \infty} \frac{|x|}{k+1} \to 0$

Using the ratio test, since $0 \leq r < 1$, this series converges.

You can see, no matter what is the value of $x$, for all $x \in \mathbb{R}$, $\lim \frac{|x|}{k+1} = 0$. Thus, this series converges for all $x \in \mathbb{R}$.

\[ \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} (x-2)^n \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \]

Method 1:

Step 1: Find $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{4^{n+1}} \cdot \frac{(-1)^n}{4^n} \right| = \lim_{n \to \infty} \left| \frac{-1}{4} \right| = \frac{1}{4} \Rightarrow L = \frac{1}{4}$

Step 2: Radius of convergence:

$R = \frac{1}{L} = \frac{1}{\frac{1}{4}} = 4 \Rightarrow$ the radius of convergence $= 4$

Thus, $R = 4$. 

Step 3: Interval of convergence

\[ |x - 2| < R \implies |x - 2| < 4 \implies -4 < x - 2 < 4 \]

By adding 2 to the inequality \(-4 < x - 2 < 4\): By adding 2 to all sides of the inequality

\[ -4 < x - 2 < 4 \]

\[ +2 \quad +2 \quad +2 \]

Thus

\[ -2 < x < 6 \]

So,

The interval of convergence is \((-2, 6)\).

Method 2: Apply the ratio test to

\[ a_n = \frac{(-1)^n}{(x - 2)^n} \]

\[ r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{(x - 2)^{n+1}}}{\frac{(-1)^n}{(x - 2)^n}} \right| = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{(x - 2)^{n+1}} \cdot 4^n}{\frac{(-1)^n}{(x - 2)^n} \cdot 4^n} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{(-1)}{1 \cdot 1 \cdot 4} \right| = \left| \frac{1}{4} \right| \left| x - 2 \right| = \frac{1}{4} \left| x - 2 \right| \]

Using the ratio test, we have a convergent series if \(|x - 2| < 1\). So, we only need to set \(r = \frac{1}{4} |x - 2| < 1\) and find the interval to have a convergent series.

\[ r = \frac{1}{4} |x - 2| < 1 \implies \left| x - 2 \right| < 4 \implies -4 < x - 2 < 4 \implies -2 < x < 6 \]

Now, the radius of convergence is the distance of \(a = 2\) to the boundary of the interval which is \(6 \text{or} -2\). \(\implies R = |6 - 2| = 4\) \(\checkmark\).
\[ \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{n!}{n!} a^n = 0 \]

**Method 1**

\[ L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n \to \infty} \frac{(n+1)!}{n!} \]

\[ \lim_{n \to \infty} (n+1) = \lim_{n \to \infty} (n+1)^\infty = \infty. \]

\[ R = \frac{1}{L} = \frac{1}{\infty} = 0 \to \text{Radius of convergence } R = 0. \]

\[ R = 0 \text{ means that the series only at } x = 0 \text{ converges.} \]

**Method 2**

Apply the ratio test to \( a_n = \frac{n!}{n!} x^n \):

\[ r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \right| x^n = \lim_{n \to \infty} \frac{(n+1)!}{n!} x^n \]

\[ \lim_{n \to \infty} (n+1)x^n = \lim_{n \to \infty} (n+1) |x| = \infty \]

By the ratio test, \( r = \infty > 1 \), the series diverges for all \( x \in \mathbb{R} \) except \( x = 0 \). As you can see, the \( \lim_{n \to \infty} (n+1)!x^n \)

is always \( \infty \) (independent of the value of \( |x| \)). So, by the ratio test, it would be diverges for all \( x \in \mathbb{R} \setminus \{0\} \).

Note that for \( x = 0 \), we have \( (at x=0) \)

\[ \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=1}^{\infty} (n!) 0^n = 0 \]

converges only at \( x = 0 \)!!!
we need to write the series as the form \( \sum C_k (x-a)^{k} \). So, we factorize 2 \((2x-6)\) to get \( \frac{(2(x-3))^{k}}{k} \). Thus, \( C_k = \frac{2^k}{k} \) and \( a = 3 \).

Find \( L = \lim_{k \to \infty} \left| \frac{C_{k+1}}{C_k} \right| = \lim_{k \to \infty} \left| \frac{2^{k+1}}{2^{k}} \right| = \lim_{k \to \infty} \left| \frac{2}{2} \right| = \lim_{k \to \infty} \left| \frac{2}{k+1} \right| \). If \( k \to \infty \), then \( \frac{2}{k+1} \) is positive, so we can remove the absolute value.

Thus, \( R = \frac{1}{L} = \frac{1}{2} \). Now, to find the interval of convergence, we solve \( |x-3| < \frac{1}{2} \). So, \(-\frac{1}{2} < x-3 < \frac{1}{2} \). By adding 2 to all sides of the inequality, we get \(-\frac{1}{2} + 3 < x-3+3 < \frac{1}{2} + 3 \). So, \( \frac{5}{2} < x < \frac{7}{2} \). Thus, the interval of convergence is \( (\frac{5}{2}, \frac{7}{2}) \).

In summary, \( R = 2 \) and the interval \( (\frac{5}{2}, \frac{7}{2}) \).

Note:
if we use method 2, we also get the same result.
(Applying the ratio test to \( a_k = \frac{2^k(x-3)^{k}}{k} \) !!!.)
\( \sum \frac{n^2 x^{2n}}{n!} = \sum \frac{2n}{n!} (x^2)^n \rightarrow \left\{ \begin{array}{l}
c_n = \frac{n^2}{n!} \\
a = 0
\end{array} \right. \)

**Method 1**

Find \( L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \to \infty} \left| \frac{(n+2)^2}{n^2} \right| = \lim_{n \to \infty} \frac{(n+2)^2}{n^2} = \lim_{n \to \infty} \frac{n^2 + 4n + 4}{n^2} = \lim_{n \to \infty} \frac{1 + \frac{4}{n} + \frac{4}{n^2}}{1} = 1. \)

So, \( L = 1 \) and then \( R = \frac{1}{\infty} = 0 \). This means the series \( \sum \frac{n^2 x^{2n}}{n!} \) converges for all \( x \in \mathbb{R} \).

**Method 2**

Apply the ratio test to \( a_n = \frac{n^2 x^{2n}}{n!} \):

\[
R = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(n+1)^2}{(n+1)!} x^{2(n+1)}}{\frac{n^2}{n!} x^{2n}} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2 x^{2n+2}}{n^2 x^{2n} (n+1)!} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2 x^2}{n^2} \right| = \lim_{n \to \infty} \left| \frac{n+1}{n^2} \right| x^2 = 0. \]

(Independent of the value of \( x \)).

Thus, since \( R = 0 \) and \( 0 < R < 1 \), by the ratio test we get that the series converges everywhere for all \( x \in \mathbb{R} \). \( \checkmark \)
1.2 Combining Power Series

The following theorem, stated without proof, gives three common ways to combine power series.

Theorem (Combining Power Series)

Suppose the power series $\sum c_n x^n$ and $\sum d_n x^n$ converge to $f(x)$ and $g(x)$, respectively, on an interval $I$.

1. **Sum and difference**: The power series $\sum (c_n \pm d_n) x^n$ converges to $f(x) \pm g(x)$ on $I$.

2. **Multiplication by a power**: Suppose $m$ is an integer such that $k+m \geq 0$ for all terms of the power series $x^m \sum c_n x^n = \sum c_n x^{n+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$. When $x=0$, the series converges to $\lim x^m f(x)$.

3. **Composition**: If $h(x) = bx^m$, where $m$ is a positive integer and $b$ is a nonzero real number, the power series $\sum c_n (h(x))^n$ converges to the composition function $f(h(x))$, for all $x$ such that $h(x)$ is in $I$. 
Example 2: Given the geometric series

\[ f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \], for \(-1 < x < 1\) \((|x| < 1)\)

Find the power series and the interval of convergence for the following functions:

I. \( \frac{x^5}{1-x} \)

We use the above formula for \( f(x) \) to find the power series for \( \frac{x^5}{1-x} \). So, we need to write \( \frac{x^5}{1-x} \) in terms of \( f(x) \).

So,

\[ \frac{x^5}{1-x} = x^5 \cdot \frac{1}{1-x} = x^5 \cdot f(x) = x^5 \cdot \sum_{n=0}^{\infty} x^n, \text{ for } -1 < x < 1 \]

\[ = \sum_{n=0}^{\infty} x^5 \cdot x^n = \sum_{n=0}^{\infty} x^{n+5}, \text{ for } -1 < x < 1 \]

Thus,

\[ \frac{x^5}{1-x} = \sum_{n=0}^{\infty} x^{n+5} = x^5 + x^6 + x^7 + x^8 + \cdots \]

and radius of convergence is \(|x| < 1\) or \(-1 < x < 1\).

Note that \( \frac{x^5}{1-x} = x^5 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + x^5 + 6 \cdot x^6 + 7 \cdot x^7 + 8 \cdot x^8 + \cdots \)

\( \leq \frac{5}{2} \leq \frac{6}{2} \leq \frac{7}{2} \leq \frac{8}{2} \)
we try to write this function in terms of \( f(x) = \frac{1}{1-x} \), and use the given formula in this example. As we mentioned, in this type of question, we try to write the function as the form \( \frac{1}{1-x} \) and use the formula. So,

\[
\frac{2}{1-5x} = 2 \cdot \frac{1}{1-5x} = 2 \cdot \frac{1}{1-(5x)} = 2 \cdot f(5x)
\]

\[
= 2 \cdot \sum_{n=0}^{\infty} (5x)^n = 2 \cdot \sum_{n=0}^{\infty} 5^n x^n = \sum_{n=0}^{\infty} 2(5^n) x^n.
\]

or

\[
\frac{2}{1-5x} = 2 \cdot f(5x) = 2 \left( 1 + (5x) + (5x)^2 + (5x)^3 + \cdots \right) = 2 \left( 1 - 5x \right)^{-1}.
\]

To use this formula we need the condition \( |5x| < 1 \) or \(-1 < 5x < 1\).

\[
\frac{2}{1-5x} = \sum_{n=0}^{\infty} 2.5^n x^n = 2 + 10x + 50x^2 + 250x^3 + \cdots
\]

Thus,

\[
\frac{2}{1-5x} = \sum_{n=0}^{\infty} 2.5^n x^n = 2 + 10x + 50x^2 + 250x^3 + \cdots
\]

for \( |5x| < 1 \).

To find radius of convergence we have \( |5x| < 1 \).

In fact, when we use the formula for \( \frac{1}{1-x} \). The thing is that we have in the denominator \( 1-0 \), has to be

\[-1 < 0 < 1 \text{ or } |10| < 1 \]. So, here it is \( 5x \):

\[|5x| < 1 \rightarrow -1 < 5x < 1 \rightarrow -\frac{1}{5} < x < \frac{1}{5} \Rightarrow \text{Radius of convergence} \]
Again we need to write it as the form \( \frac{1}{1-x} \), but here we have \( 1+x^2 \) in the denominator. To have negative sign behind \( x^2 \), we write \( +(-1)(-1) \). So, we have

\[
1 + x^2 = 1 - (-x^2) .
\]

Now, we replace \( x^2+1 \) by \( 1-(-x^2) \), we get

\[
\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} .
\]

or

\[
\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 + (-x^2) + (-x^4) + (-x^6) + \ldots
\]

\[
= 1 - x^2 + x^4 - x^6 + x^8 + \ldots
\]

Thus,

\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + \ldots
\]

**Radius of convergence:**

\[
\frac{1}{1-(-x^2)} \quad \text{and} \quad \frac{1}{1-(-x^2)} < 1 \quad \Rightarrow \quad 1 - x^2 | < 1 \quad \Rightarrow \quad |x^2| < 1 \quad \Rightarrow \quad 1 \leq |x| < 1
\]

\( \text{interval of convergence} \)
Let write it as the form \( \frac{1}{1 - (-5x^2)} \)

\[
\frac{3x^3}{1 + 25x^2} = \frac{\frac{3x^3}{1 - (-25x^2)}}{1 - (-5x^2)} = 3x^3 \cdot \frac{1}{1 - (-5x^2)} = 3x^3 \cdot \sum_{n=0}^{\infty} (-5x^2)^n
\]

\[
= 3x^3 \sum_{n=0}^{\infty} (-1)^n \cdot (5^2 \cdot x^2)^n = 3 \sum_{n=0}^{\infty} (-1)^n \cdot 5^{2n} \cdot x^{2n}
\]

\[
= 3 \sum_{n=0}^{\infty} (-1)^n \cdot 5^{2n} \cdot x^{2n+3}
\]

Thus,

\[
\frac{3x^3}{1 + 25x^2} = \sum_{n=0}^{\infty} 3(-1)^n \cdot 5^{2n} \cdot x^{2n+3}
\]

3 \( x^3 - 3(5^2) \cdot x^5 + 3 \cdot 5^4 \cdot x^7 - 3 \cdot 5^6 \cdot x^9 + \ldots \)

To write it as the form \( c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n \)

we have:

\[
\frac{0 \cdot x^0}{c_0} = \frac{0 \cdot x^1}{c_1} = \frac{3 \cdot 5^2}{c_2} = \frac{0 \cdot x^4}{c_3} = \frac{3(5^2)^2}{c_4} = \frac{0 \cdot x^6}{c_5} \ldots
\]

Radius of convergence:

\[
\left| -5x^2 \right| < 1 \quad \Rightarrow \quad \left| 5^2 x^2 \right| < 1 \quad \Rightarrow \quad 15x^2 < 1
\]

\[
\Rightarrow -1 < 5x < 1 \quad \Rightarrow \quad -\frac{1}{5} < x < \frac{1}{5}
\]