9.3 Partial Fractions (Long Division)

Here, we deal with the integral of form \( \int \frac{f(x)}{q(x)} \, dx \), where \( f(x) = \frac{p(x)}{q(x)} \), and \( p \) and \( q \) are polynomials, and also \( \text{degree } p(x) > \text{degree } q(x) \).

In this case, since degree of \( p(x) \) is greater than degree of \( q(x) \), we apply "long division" to write \( \frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)} \). Now, in \( \frac{r(x)}{q(x)} \), degree of \( r(x) \) < degree of \( q(x) \). So, we can deal with \( \frac{r(x)}{q(x)} \), using the methods that we learned in sections 9.1 & 9.2!!

In summary:

\[
\int f(x) \, dx = \int \frac{p(x)}{q(x)} \, dx
\]

Perform "Long Division":

\[
\begin{align*}
\frac{p(x)}{q(x)} &= \frac{h(x)}{q(x)} \quad \text{(degree } p(x) > \text{degree } q(x)) \\
&= h(x) + \frac{r(x)}{q(x)} \\
\end{align*}
\]

Therefore,

\[
\int f(x) \, dx = \int \frac{p(x)}{q(x)} \, dx = \int \left[ h(x) + \frac{r(x)}{q(x)} \right] \, dx
\]

\[
= \int h(x) \, dx + \int \frac{r(x)}{q(x)} \, dx
\]

polynomial; easy to integrate

apply partial fractions in 9.1 & 9.2
Example 1: \[ \int \frac{3x^3 + x^2 - 2x + 10}{x^2 + x - 2} \, dx \]

\[ \text{degree is 3} \]

\[ \text{degree is 2} \]

So, by using the "Long Division", we get:

\[
\begin{array}{c|ccccc}
& 3x^3 & + & x^2 & - 2x & + 10 \\
\hline
x^2 & + & x & - 2 & | & 3x^2 \\
- & (3x^3 & + & 3x^2 & - 6x) & \\
\hline
& -2x^2 & + & 4x & + 10 \\
- & (-2x^2 & + & 2x & + 4) & \\
\hline
& 6x & + 6 \\
\end{array}
\]

Now, we can rewrite the integrand as:

\[
3x^3 + x^2 - 2x + 10 \quad \text{over} \quad x^2 + x - 2 = 3x - 2 + \frac{6x + 6}{x^2 + x - 2}
\]

degree 1

polyomial; easy to integrate

degree 2

Let's use method of partial fraction to write \( \frac{6x + 6}{x^2 + x - 2} \) as:

\[
\frac{6x + 6}{x^2 + x - 2} = \frac{A}{x + 2} + \frac{B}{x - 1}
\]

Note that \( x^2 + x - 2 = (x + 2)(x - 1) \). So,

\[
\frac{6x + 6}{x^2 + x - 2} = \frac{6x + 6}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1}
\]
By multiplying \((x+2)(x-1)\) into both sides of the last equality, we get

\[6x + 6 = A(x-1) + B(x+2)\]

Roots of \((x+2)\) and \((x-1)\) are \(x = -2\) and \(x = 1\). So,

\[x = 1 \text{ in } (\star): \quad 6(1) + 6 = A(0) + B(3) \Rightarrow 12 = 3B \Rightarrow B = 4\]

\[x = -2 \text{ in } (\star): \quad 6(-2) + 6 = A(-3) + B(0) \Rightarrow -6 = -3A \Rightarrow A = 2\]

Therefore,

\[\frac{6x + 6}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{4}{x-1}\]

and

\[\frac{3x^3 + x^2 - 2x + 10}{x^2 + x - 2} = 3x - 2 + \frac{2}{x+2} + \frac{4}{x-1}\]

Finally,

\[
\int \frac{3x^3 + x^2 - 2x + 10}{x^2 + x - 2} \, dx = \int (3x - 2) \, dx + \int \frac{2}{x+2} \, dx + \int \frac{4}{x-1} \, dx
\]

\[
= \frac{3x^2}{2} - 2x + 2 \ln |x+2| + 4 \ln |x-1| + C
\]

\[
= \frac{3}{2} x^2 - 2x + 2 \ln |x+2| + 4 \ln |x-1| + C
\]
Example 2: Evaluate \( \int \frac{x^4+2x+7}{x^2+3x+2} \, dx \) \( \rightarrow \) degree is 4 \( \rightarrow \) degree is 2

So, we use "Long Division" to get

\[
\begin{align*}
\frac{x^4+2x+7}{x^2+3x+2} &= \left( \frac{x^2-3x+7}{x^2+3x+2} \right) - (x^4+3x^3+2x^2) \\
&
\phantom{=} -3x^3 - 9x^2 - 6x \\
&
\phantom{=} \frac{7x^2 + 8x + 7}{x^2+3x+2} \\
&
\phantom{=} -(7x^2 + 21x + 14) \\
&
\phantom{=} -13x - 7
\end{align*}
\]

Therefore,
\[
\frac{x^4+2x+7}{x^2+3x+2} = x^2 - 3x + \frac{-13x - 7}{x^2+3x+2}
\]

By long division

**Step 2**: Partial Fractions

Decomposition to \( \frac{13x+7}{x^2+3x+2} \).

\[
\frac{13x+7}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} \quad (I)
\]

Multiplying \((I)\) by the denominator \((x+1)(x+2)\), we get

\[
13x+7 = A(x+2) + B(x+1) \quad (\star)
\]

The roots of \((x+1)\) & \((x+2)\) are \(x=-1\) and \(x=-2\).

Let \(x=-1\) in \((\star)\):

\[
-13 + 7 = A(1) + B(0) \quad \Rightarrow \quad A = -6
\]

Let \(x=-2\) in \((\star)\):

\[
-26 + 7 = A(0) + B(-1) \quad \Rightarrow \quad -19 = -B \quad \Rightarrow \quad B = 19
\]

**Step 3** Evaluate the integral using step 1 & 2. By step 1 and 2, we can rewrite \( \frac{x^4+2x+7}{x^2+3x+2} \) as:

\[
\frac{x^4+2x+7}{x^2+3x+2} = x^2 - 3x + \left( -\frac{6}{x+1} + \frac{19}{x+2} \right) = x^2 - 3x + 7 + \frac{6}{x+1} - \frac{19}{x+2}
\]

Thus,

\[
\int \frac{x^4+2x+7}{x^2+3x+2} \, dx = \int (x^2 - 3x + 7) \, dx + 6 \int \frac{dx}{x+1} - 19 \int \frac{dx}{x+2} = \frac{3}{2} x - \frac{3}{2} x^2 + 7 x + 6 \ln |x+1| - 19 \ln |x+2| + C
\]