Lecture note 15
Feb 5, 2016

5. Substitution Rule

How the Substitution Rule help us to compute an Indefinite Definite Integral? Let’s explain it with the following examples:

Example 1. (I) \( \int 6x^2 (2x^3 - 5)^6 \, dx \)

If we choose \( u = 2x^3 - 5 \), then we have its derivative, \( 6x^2 \), in our integral. So, it is a good option.

\[ u = 2x^3 - 5 \quad \Rightarrow \quad du = 6x^2 \, dx \]

Substitute to integral:

\[ \int 6x^2 (u)^6 \, \frac{du}{6x^2} = \int u^6 \, du \]

\[ = \frac{u^{11}}{11} + C = \frac{(2x^3 - 5)^{11}}{11} + C \]

Example 2. \( \int_0^3 \frac{2x}{\sqrt{x^2 + 16}} \, dx \)

Since we have \( \sqrt{x^2 + 16} \) in the denominator, and the derivative of \( x^2 + 16 \) on top, which is \( 2x \), so, the best option would be \( u = x^2 + 16 \). Then, \( du = 2x \, dx \).

\[ \int_0^3 \frac{2x}{\sqrt{x^2 + 16}} \, dx = \int \frac{25}{16} \, du = \frac{25}{16} \int_0^3 \frac{du}{\sqrt{u}} \]

\[ = \left. \frac{25}{16} \sqrt{u} \right|_{u=x^2+16}^{u=25} = 2\sqrt{25} - 2\sqrt{16} = 2(5) - 2(4) = 10 - 8 = 2 \]
**Substitution Rule for**

1. **Indefinite Integral**
   - Let \( u = g(x) \), where \( g' \) is continuous on an interval, and let \( f \) be continuous on that interval. On that interval:
   \[
   \int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du
   \]

Procedure: **substitution rule**

"Change of variables."

1. **Indefinite Integral**
   - **Step 1:** Figure out the best option for substitution \( u = g(x) \).
   - **Step 2:** \( du = g'(x) \, dx \)
   - **Step 3:** Substitute \( u = g(x) \) and \( du = g'(x) \, dx \) in the integral
   - **Step 4:** Evaluate the new integral with respect to \( u \)
   - **Step 5:** Write the result in terms of \( x \) using \( u = g(x) \). [Do not forget \( + C \), constant]

2. **Definite Integral**
   - Let \( u = g(x) \), where \( g' \) is continuous on \([a, b]\), and let \( f \) be continuous on the range of \( g \). Then
   \[
   \int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
   \]

**Step 1:** Figure out the best option for substitution \( u = g(x) \).
**Step 2:** Substitute \( u = g(x) \) and \( du = g'(x) \, dx \) in the integral
**Step 3:** Find the new upper & lower bound for the new integral by \( g(a) \) [lower] and \( g(b) \) [upper]
**Step 4:** Evaluate the new integral with respect to \( u \)
**Step 5:** Write the result, which is a value (a real number)
Now, we use the substitution rule to compute the following examples:

Example 1: compute \( u = -3x \rightarrow du = -3dx \rightarrow dx = \frac{du}{-3} \)

(I) \( \int e^{-3x} \, dx = \int e^u \frac{du}{-3} = \frac{-1}{3} \int e^u \, du = -\frac{e^u}{3} + C = -\frac{e^{-3x}}{3} + C \)

(II) \( \int \frac{1}{5x-1} \, dx \)

let \( u = 5x - 1 \)
\( du = (5x - 1)' \, dx = 5 \, dx \)

\( \int \frac{1}{u} \, du = \frac{1}{5} \int \frac{du}{u} = \frac{1}{5} \ln|u| + C \)

Variations on the substitution method

Example 2: compute \( \int \frac{x}{\sqrt{x-3}} \, dx \)

The best option for substitution is \( u = x - 3 \). Then, \( du = (x-3)' \, dx = 1 \cdot dx = dx \).

So, we get that \( \int \frac{x}{\sqrt{x-3}} \, dx = \int \frac{x}{\sqrt{u}} \, du \). Now, we need to find \( x \) in terms of \( u \), and substitute it in the above integral. [Note that after substitution the new integral has to be in terms of (only) \( u \).] So, we use \( u = x - 3 \). So, we have \( x = u + 3 \). Now, plugging \( x = u + 3 \) into \( \int \frac{x}{\sqrt{u}} \, du \), we get

\[ \int \frac{x}{\sqrt{x-3}} \, dx = \int \frac{x}{\sqrt{u}} \, du = \int \frac{u + 3}{\sqrt{u}} \, du = \int \left( \frac{u}{\sqrt{u}} + \frac{3}{\sqrt{u}} \right) \, du = \int \left( \frac{u}{u^{1/2}} + \frac{3}{u^{1/2}} \right) \, du = \int u^{1/2} \, du + 3 \int u^{-1/2} \, du = \int u^{1/2} \, du + 3 \int u^{-1/2} \, du = \frac{2}{3} u^{3/2} + 3 u^{1/2} + C \]

Example 3: compute \( \int_0^\frac{\pi}{2} \sin^4 x \cos x \, dx \)

\( \sin^4 x \) has power 4, and the derivative of \( \sin x \), is \( \cos x \), and we have it in our integral. So, the best option for substitution is \( u = \sin x \). Then \( du = \cos x \, dx \). Therefore, since \( x \) is changing from 0 to \( \frac{\pi}{2} \), \( u = \sin x \) is changing from \( \sin(0) = 0 \) to \( \sin \frac{\pi}{2} = 1 \)

\[ \int_0^\frac{\pi}{2} \sin^4 x \cos x \, dx = \int_0^1 \frac{u^4}{u} \, du = \int_0^1 u^4 \, du = \frac{u^5}{5} \bigg|_0^1 = \frac{1}{5} - \frac{0}{5} = \frac{1}{5} \]
Example 4: Compute \( \int x^5 (-3x^6 + 7)^2 \, dx \). Let \( u = -3x^6 + 7 \). So, 
\[ du = (-3x^6 + 7) \, dx = -18x^5 \, dx \], and 
\[ dx = \frac{du}{-18x^5} \]. Substitute in the integral:
\[ \int x^5 (-3x^6 + 7)^2 \, dx = \int x^5 \frac{u^2 \, du}{-18x^5} = -\frac{1}{18} \int u^2 \, du = -\frac{1}{18} \frac{u^3}{3} + C \]
\[ = -\frac{1}{378} \frac{21}{378} u + C = -\frac{1}{378} (-3x^6 + 7) + C \]

Example 5. Evaluate \( \int_0^\frac{\pi}{2} \cos^2 \theta \, d\theta \) [Integral \( \cos^2 \theta \) & \( \sin^2 \theta \)]

**Hint:** Use the formula \( \cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \)

\[ \int_0^\frac{\pi}{2} \cos^2 \theta \, d\theta = \int_0^\frac{\pi}{2} \frac{1 + \cos(2\theta)}{2} \, d\theta = \frac{1}{2} \int_0^\frac{\pi}{2} d\theta + \int_0^\frac{\pi}{2} \cos(2\theta) \, d\theta \]

\[ = \frac{1}{2} \theta \bigg|_0^\frac{\pi}{2} + \frac{1}{2} \int_0^\frac{\pi}{2} \cos(2\theta) \, d\theta \]

**Substitution rule**

\[ \frac{1}{2} \int_0^\frac{\pi}{2} \cos(2\theta) \, d\theta = \frac{1}{2} \left[ \sin(2\theta) \right]_0^\frac{\pi}{2} \]

\[ = \frac{1}{2} \left( \frac{\pi}{2} - 0 \right) + \frac{1}{2} (0) = \frac{\pi}{4} \]

Example 6: Evaluate \( \int \frac{(\sqrt{x} + 1)^4}{2\sqrt{x}} \, dx \)

Since \( (\sqrt{x} + 1) \) has power 4 and the derivative of \( (\sqrt{x} + 1) \) is \( \frac{1}{2\sqrt{x}} \), and we have it in the denominator, the best option for substitution is \( u = \sqrt{x} + 1 \). So, 
\[ du = \frac{1}{2\sqrt{x}} \, dx \]. Therefore,
\[ \int \frac{(\sqrt{x} + 1)^4}{2\sqrt{x}} \, dx = \int u^4 \, du = \frac{u^5}{5} + C = \frac{(\sqrt{x} + 1)^5}{5} + C \]

\[ \int \frac{(\sqrt{x} + 1)^4}{2\sqrt{x}} \, dx = \int \frac{u^4}{2\sqrt{x}} \, (2\sqrt{x} \, du) = \int u^4 \, du! \]