Review: [Derivative of Integrals]

Let $f$ be continuous, $g(x)$ and $h(x)$ be differentiable, then

$$\frac{d}{dx} \int_{h(x)}^{g(x)} f(t) \, dt = f(g(x)) \frac{d}{dx} g(x) - f(h(x)) \frac{d}{dx} h(x)$$

**Example 1.** Compute $\frac{d}{dx} \int_{10}^{x^2} \frac{du}{u^4 + 1}$ using the above formula, we get

$$\frac{d}{dx} \int_{10}^{x^2} \frac{du}{u^4 + 1} = \frac{1}{(10)^4 + 1} \cdot (10) - \frac{1}{(x^2)^4 + 1} \cdot (x^2)' = 0 - \frac{1}{x^8 + 1} \cdot (2x)$$

$$= \frac{-2x}{x^8 + 1}$$

**Example 2.** Compute $\frac{d}{dx} \int_{x-1}^{e^{2x}} \frac{\sin(t)}{t} \, dt$. We use the formula to get

$$\frac{d}{dx} \int_{x-1}^{e^{2x}} \frac{\sin(t)}{t} \, dt = \frac{\sin(e^{2x})}{e^{2x}} \cdot (e^{2x})' - \frac{\sin(x-1)}{x-1} \cdot (x-1)'$$

$$= \frac{\sin(e^{2x})}{e^{2x}} \cdot 2 e^{2x} - \frac{\sin(x-1)}{x-1} \cdot (1)$$

$$= 2 \sin(e^{2x}) - \frac{\sin(x-1)}{x-1}$$
4. Antiderivative and Indefinite Integrals

Given Function $f$ To find $f'$

Goal of Differentiation $f$

Goal of Antiderivative $f'$

we already know how to deal with differentiation, so to deal with Antiderivative of $f' = F$, we only need to think **backward**.

**Definition (Antiderivative)**

A function $F$ is an antiderivative of $f$ on $[a, b]$ if

$$F(x) = f(x), \text{ for all } x \in [a, b].$$

Example 3:

(I) $\frac{d}{dx}(x^4) = 4x^3 \implies$ an antiderivative of $f(x) = 4x^3$ is $F(x) = x^4$

(II) $\frac{d}{dx}(x^4 - 10) = 4x^3 \implies$ an antiderivative of $f(x) = 4x^3$ is $F(x) = x^4 - 10$

So, $x^4$ and $x^4 - 10$ are the antiderivatives of $f(x) = 4x^3$.

Example 4: Find all the antiderivatives of $f(x) = \frac{1}{1 + x^2}$

we first need to an antiderivative of $f(x) = \frac{1}{1 + x^2}$. we know that

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1 + x^2} \implies \tan^{-1}(x) \text{ is antiderivative of } \frac{1}{1 + x^2}. \text{ But, we check } \tan^{-1}(x) + 1, \tan^{-1}(x) - 10, \tan^{-1}(x) + 1000, \text{ they are also antiderivatives of } \frac{1}{1 + x^2}. \text{ So, in general, } \tan^{-1}(x) + C \text{ (where } C \text{ is an arbitrary constant) are all antiderivatives.}
Let $F$ be any antiderivative of $f$. Then, all the antiderivatives of $f$ have the form $F + C$, where $C$ is an arbitrary constant.

If $F(x)$ is any antiderivative of $f(x)$, all the antiderivatives $F(x) + C$ are vertical translations of one another—they differ by a constant.

Reason: All of them have the same derivative, which is $f(x)$. So, they have the same slope at each point. So, we only have vertical shift.

**Definition (Indefinite Integrals)**

Let $F(x)$ be an antiderivative of $f(x)$, that is, $F'(x) = f(x)$. The Indefinite Integral

$$\int f(x) \, dx = F(x) + C,$$

means find all the antiderivatives of $f(x)$.

"an arbitrary constant"

For example, if you take a look at example 4, we found all antiderivative of $f(x) = \frac{1}{1+ x^2}$. Thus,

$$\int \frac{1}{1+ x^2} \, dx = \tan^{-1}(x) + C.$$

**Example 5.** Find the indefinite integral

$$\int (\sqrt{x} + e^{2x} - \sec^2(x)) \, dx =$$

$$\int \sqrt{x} \, dx + \int e^{2x} \, dx - \int \sec^2(x) \, dx =$$

antiderivative

$$\frac{2}{3} x^{\frac{3}{2}} + \frac{1}{2} e^{2x} - \tan(x) + C$$

antiderivative

$$d\left(\tan(x)\right) = \sec(x)$$

Our guess

$$\frac{d}{dx}\left(\frac{2}{3} x^{\frac{3}{2}}\right) = \frac{1}{2} x^{\frac{1}{2}} \rightarrow 2x^{\frac{1}{2}} \text{ is one way to cancel } \frac{3}{2}.$$

$$\frac{d}{dx}\left(\frac{e^{2x}}{2}\right) = e^{2x} \rightarrow \frac{1}{2} e^{2x} \text{ is another way to cancel } 2.$$
Examples \[ \text{Definite and Indefinite Integrals} \]

Procedure:

\{ \text{Antiderivative} + C \} \rightarrow \text{Indefinite Integral}

\{ \text{Antiderivative} \}

\{ + \text{Fundamental Theorem} \} \rightarrow \text{Definite Integral}

Example 6: Compute \( \int \frac{3\sqrt{t}-t}{t^2} \, dt \) [indefinite integral] when you see this form integral (involving polynomial), break it into two (sometimes more than two) parts. But how? We can write \( \frac{3\sqrt{t}-t}{t^2} \) as \( \frac{3\sqrt{t}}{t^2} - \frac{t}{t^2} \), and simplify \( \frac{3\sqrt{t}}{t^2} = \frac{\sqrt{t}}{t} = \frac{t^{1/2}}{t} = t^{-1/2} \) and \( \frac{t}{t^2} = \frac{1}{t} = t^{-1} \). So,

\[
\int \frac{3\sqrt{t}-t}{t^2} \, dt = \int \frac{\sqrt{t}}{t^2} \, dt - \int \frac{1}{t} \, dt = \int t^{-1/2} \, dt - \int t^{-1} \, dt = -2t^{1/2} - \ln|t| + C
\]

You can also use Table

\[
\int t^{\frac{5}{3}} \, dt = \frac{t^{8/3}}{\frac{8}{3}} + C = \frac{3}{8}t^{8/3} + C
\]

Example 7: Compute \( \int_{\pi/4}^{2\pi} \cos\left(\frac{X}{4}\right) \, dX \)

The antiderivative of \( \cos\left(\frac{X}{4}\right) \) is \( 4 \sin\left(\frac{X}{4}\right) \)

\[
\int_{\pi/4}^{2\pi} \cos\left(\frac{X}{4}\right) \, dX = 4 \sin\left(\frac{2\pi}{4}\right) - 4 \sin\left(\frac{\pi}{4}\right) = 4(1) - 4\left(\frac{\sqrt{2}}{2}\right) = 4 - 2\sqrt{2}
\]

Example 8: Compute \( \int_0^1 (x-2)(1-3x) \, dx \)

To deal with this type integral, we compute and write it as a polynomial, then take the integral one by one. We have \( (x-2)(1-3x) = x-3x^2-2+6x = -3x^2+7x-2 \). So,

\[
\int_0^1 (x-2)(1-3x) \, dx = \int_0^1 (-3x^2+7x-2) \, dx = \int_0^1 -3x^2 \, dx + \int_0^1 7x \, dx - \int_0^1 2 \, dx = \left[ -x^3 \right]_0^1 + \left[ \frac{7x^2}{2} \right]_0^1 - 2(x)_0^1 = (1-0)^3 + \left( \frac{7(1)^2}{2} - \frac{7(0)^2}{2} \right) - \left( 2(1) - 2(0) \right) = \frac{1}{2}
\]
Example 9: Compute \( \int \frac{-7}{\sqrt{4-x^2}} \, dx = -7 \int \frac{dx}{\sqrt{2^2-x^2}} = -7 \sin^{-1} \left( \frac{x}{2} \right) + C \)

\[ \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C \]

**Integral of Even and Odd Functions**

Let \( f \) be integrable on \([-a,a]\), then

1. \( f(x) \) even \( \rightarrow \) \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \)
2. \( f(x) \) odd \( \rightarrow \) \( \int_{-a}^{a} f(x) \, dx = 0 \)

Example 10. Compute \( \int_{-3}^{3} x^3 \, dx = 0 \)

\( f(x) = x^3 \)
\( f(-x) = (-x)^3 = -x^3 \)
\( f(-x) \) even, if \( f(-x) = f(x) \), for all \( x \in [-a,a] \)

Example 11. Compute \( \int_{-1}^{1} (4x^2 + x^4) \, dx \)

\( f(x) = x^4 + x^2 \)
\( f(-x) = (-x)^2 + (-x)^4 = x^2 + x^4 \)

\( f(x) \) even, \( \Rightarrow \) \( f(-x) = f(x) \)

\( \int_{-1}^{1} (4x^2 + x^4) \, dx = 2 \int_{0}^{1} (4x^2 + x^4) \, dx = 2 \left( \frac{x^5}{5} + \frac{x^3}{3} \right) \bigg|_{0}^{1} = 2 \left( \frac{1}{5} + \frac{1}{3} \right) = \frac{16}{15} \)

Example 12. Compute \( \int_{-\pi/2}^{\pi/2} (\sin(x) - 2\cos(x)) \, dx \)

\( = \int_{-\pi/2}^{\pi/2} \sin(x) \, dx - \int_{-\pi/2}^{\pi/2} 2\cos(x) \, dx \)

\( = -2(2) \int_{0}^{\pi/2} \cos(x) \, dx = -4 \int_{0}^{\pi/2} \cos(x) \, dx = -4 \sin(x) \bigg|_{0}^{\pi/2} = -4 \sin(\pi/2) - (-4 \sin(0)) = -4 \)
Substitution Rule !!!

Let's take a look at the integral $\int (2x-3)^6 \, dx$. Using the tools and techniques that we know, we only can expand $(2x-3)^6$ and then take integral one by one, which is difficult and tedious. So, we need to find another (good) way to deal with this. We know if we just have $\int u^6 \, du$, then we can compute it easily. So, let's write $\int (2x-3)^6 \, dx$ as this form ($\int u^6 \, du$). How can we do it?

Answer: Just let $u = 2x - 3$, then we'll have $\int u^6 \, dx$. But, we also need to change $dx$ to $du$. [Because when our function is in terms of $u$, we have to have $du$ for integral sign !!!]

How can we write $dx$ in terms of $du$? We just have one option, which is using the relation between them $u = 2x - 3$.

Take $d$ from both sides $\Rightarrow \; du = d(2x-3) = (2x-3)' \, dx = 2 \, dx$

So, $\int \frac{du}{2}$. Now plug $dx = \frac{du}{2}$ into $\int u^6 \, dx$ to get everything in terms of $u$. $\Rightarrow \; \frac{1}{2} \int u^6 \, du = \frac{1}{2} \int u^6 \, du$, so,

$$\int (2x-3)^6 \, dx \quad \overset{\text{Step 1}}{\Rightarrow} \quad \frac{1}{2} \int u^6 \, du = \frac{1}{2} \frac{u^7}{7} + C$$

$$\overset{\text{Step 2}}{=} \quad \frac{u^7}{14} + C$$

Take this integral (much easier than first one) plugging back $u = 2x - 3$ to final answer $\frac{u^7}{14} + C$. 
Indefinite Integral (Table)

1. \[ \int x^p \, dx = \frac{x^{p+1}}{p+1} + C, \quad p \neq -1 \]
2. \[ \int \frac{1}{x} \, dx = \ln |x| + C \quad (x \neq 0) \]
3. \[ \int e^{ax} \, dx = \frac{1}{a} e^{ax} + C \]
4. \[ \int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C \]
5. \[ \int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C \]
6. \[ \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + C \]
7. \[ \int \csc^2(ax) \, dx = -\frac{1}{a} \cot(ax) + C \]
8. \[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C \quad (for \ |x| \leq |a|, \ a > 0) \]
9. \[ \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \quad (for \ all \ x \ and \ a \neq 0) \]