Lecture note 10
Jan 25, 2016

Integration

the tangent lines to curves ➔ Derivative
the area underneath curves ➔ Integral

1. Approximating Areas by Riemman Sums

First, we should recall that we know how to compute the following area:

![Area formulas for various shapes]

but what if we want to compute this area:

![Graph with shaded area]

Goal and Idea:

1. Approximate the area using rectangles!
2. Better and better approximations to get smaller error!
Goal and Idea 1: Approximate the area using rectangles.

We'll consider three different cases:

- **Case one [Left Endpoints]**

\[ f(x_1) = f(2) \]
\[ f(x_2) = f(3) \]
\[ f(x_3) = f(4) \]
\[ f(x_4) = f(5) \]

we divide the interval \([a, b] = [2, 6]\) into four sub-intervals. What is the length of each sub-interval?

\[ a = 2 \]
\[ b = 6 \]

length of each sub-interval = \( \frac{b - a}{4} = \frac{6 - 2}{4} = 1 \)

Sum of the areas of the rectangles = \( A_1 + A_2 + A_3 + A_4 \)
\[ = f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1 \]
\[ < \text{Area under the graph} \]

- **Case two [Right Endpoints]**

- **Case three [Midpoints]**

Sum of the areas of the rectangles > Area under the graph
2) **Goal and Idea 2** Better approximations to get Smaller error.

we want to make the approximation better, so, we use more, thinner rectangles in all cases (one, two, and three). This means we divide the interval \([a,b]\) to more intervals with smaller lengths.

\[
\text{area of each rectangle} = f(x_i) \cdot \Delta x \quad \text{for } i = 1, 2, \ldots, n.
\]

\[
\text{height of rectangle} \rightarrow \text{length of each interval} = \frac{b-a}{n}
\]

Sum of areas of the rectangles: 
\[
f(x_1) \Delta x + f(x_2) \Delta x + \ldots + f(x_i) \Delta x + \ldots + f(x_n) \Delta x
\]

since we used the right endpoint, this sum is called **Right Riemann Sum**. 

→ \( R_n \)

NOW we can see if we use more rectangle, which mean we choose \( n \) larger and larger, we'll get better and better approximation.

If we use left endpoint the sum is called **Left Riemann Sum**. 

→ \( L_n \)

Finally, we use midpoint the sum 
\[
f(x_1) \Delta x + f(x_2) \Delta x + \ldots + f(x_n) \Delta x
\]

is called **Midpoint Riemann Sum**. 

→ \( M_n \)
Definition [Regular Partition]
Suppose \([a, b]\) is a closed interval containing \(n\) sub-intervals
\([x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\)
of equal length \(\Delta x = \frac{b-a}{n}\) with \(a = x_0\) and \(b = x_n\). The endpoints
\(x_0, x_1, x_2, \ldots, x_{n-1}, x_n\) of the sub-intervals are called grid points, and they
create a regular partition of the interval \([a, b]\). In general, the
\(i\)th grid point is \(x_i = a + i \Delta x\), for \(i = 0, 1, 2, \ldots, n\).

\[\Delta x = \frac{b-a}{n}\]

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Definition [Riemann Sum]
Suppose \(f\) is defined on a closed interval \([a, b]\), which is divided into
\(n\) sub-intervals of equal length \(\Delta x\). If \(x^*_i\) is any point in the \(i\)th
sub-interval \([x_{i-1}, x_i]\), for \(i = 1, 2, \ldots, n\), then

\[
\sum_{i=1}^{n} f(x^*_i) \Delta x + \sum_{i=1}^{n} f(x^*_i) \Delta x + \ldots + f(x^*_n) \Delta x
\]

is called "Riemann Sum" for \(f\) on \([a, b]\). This sum is called

1. a left Riemann Sum if \(x^*_i\) is the left endpoint of \([x_{i-1}, x_i]\)
2. a right Riemann Sum if \(x^*_i\) is the right endpoint of \([x_{i-1}, x_i]\)
3. a midpoint Riemann Sum if \(x^*_i\) is the midpoint of \([x_{i-1}, x_i]\),
for \(i = 1, 2, \ldots, n\).
Example 1. Let $R$ be the region bounded by the graph of $f(x) = x^2$ between $x=1$ and $x=3$. Estimate the area using four approximating rectangles and

(i) right endpoint ($R_4$)

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$R_4 = \text{sum of areas of rectangles (using right endpoint)}$

$= f(\frac{3}{2}) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f(\frac{5}{2}) \cdot \frac{1}{2} + f(3) \cdot \frac{1}{2}$

$= \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} + (3)^2 \cdot \frac{1}{2} = \frac{43}{4}$

(ii) left endpoint ($L_4$)

$L_4 = \text{sum of areas of rectangles (using left endpoint)}$

$= f(1) \cdot \frac{1}{2} + f(\frac{3}{2}) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f(\frac{5}{2}) \cdot \frac{1}{2}$

$= (1)^2 \cdot \frac{1}{2} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2}$

$= \frac{27}{4}$

(iii) midpoint ($M_4$)

$M_4 = \text{sum of areas of rectangles (using midpoint)}$

$= f(\frac{5}{4}) \cdot \frac{1}{2} + f(\frac{7}{4}) \cdot \frac{1}{2} + f(\frac{9}{4}) \cdot \frac{1}{2} + f(\frac{11}{4}) \cdot \frac{1}{2}$

$= \left(\frac{5}{4}\right)^2 \cdot \frac{1}{2} + \left(\frac{7}{4}\right)^2 \cdot \frac{1}{2} + \left(\frac{9}{4}\right)^2 \cdot \frac{1}{2} + \left(\frac{11}{4}\right)^2 \cdot \frac{1}{2}$

$= \frac{69}{16}$

midpoint of $1$ and $\frac{3}{2} = \frac{1 + \frac{3}{2}}{2} = \frac{5}{4}$
Working with Riemann Sum is cumbersome with large numbers of sub-intervals. In fact, we'll be dealing with sums of areas of many rectangles \( f(x_1^*) \Delta x + f(x_2^*) \Delta x + \ldots + f(x_n^*) \Delta x \).

So, to avoid to write this summation again and again, we define a notation to express sums in a compact way.

Sigma (Summation) Notation

"upper limit of summation"

"Summation index"

"lower limit of summation"

"depends on i"

Examples:

\[ 1 + 2 + 3 + \ldots + 15 = \sum_{i=1}^{15} i \]

\[ 0 + 1 + 2 + \ldots + 9 = \sum_{i=0}^{9} i \]

\[ \sum_{i=2}^{10} (2i+1) = [2(2)+1] + [2(3)+1] + \ldots + [2(10)+1] \]

\[ \sum_{i=1}^{20} f(i) = f(1) + f(2) + f(3) + \ldots + f(20) \]

It can be any function of \( i \), like \( \sin(i) \), \( \ln(i) \), \( i^3 + 2i + 1 \).
Note: The index in a sum is a "dummy variable". It means that it doesn't matter what symbol you choose as an index.

Example:
\[
\sum_{k=1}^{q_0} k = \sum_{i=1}^{q_0} i = \sum_{m=1}^{q_0} m = 1 + 2 + 3 + \ldots + q_0.
\]

Example 2. Evaluate the following expressions:

(I) \[
\sum_{i=1}^{4} \frac{3}{i^3} = 1^3 + 2^3 + 3^3 + 4^3.
\]

(II) \[
\sum_{n=0}^{3} \frac{1}{n^2 + 1} = \frac{1}{0^2 + 1} + \frac{1}{1^2 + 1} + \frac{1}{2^2 + 1} + \frac{1}{3^2 + 1}
\]

Example 3: Express the following sums using sigma notation.

(I) \[
\sum_{i=10}^{14} i
\]

(II) \[
\sum_{i=0}^{3} (4 + 2i)
\]

(III) \[
\sum_{i=1}^{5} \frac{1}{i}
\]

Pattern is \[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}
\]
Suppose that \( \{a_1, a_2, \ldots, a_n\} \) and \( \{b_1, b_2, \ldots, b_n\} \) are two sets of real numbers, and \( c \) is a real number. Then,

(I) \[ \sum_{i=1}^{n} c \cdot a_i = c \sum_{i=1}^{n} a_i \]

(II) \[ \sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \]

(III) \[ \sum_{i=1}^{n} a_i - b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \]

**Sum of Powers Integers**

**Special Sums you should know!!!**

Let \( n \) be a positive integer and \( c \) a real number.

(I) \[ \sum_{i=1}^{n} c = c \cdot n \]

(II) \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

(III) \[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

(IV) \[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \]