Dot Products

The $xyz$-coordinate System:

By adding a new axis (the $z$-axis) to the $xy$-coordinate system, we create the following $xyz$-coordinate system:

\[(0,0)\]
So,

when $z = 0$ we are on the $xy$-plane

when $x = 0$ we are on the $yz$-plane

when $y = 0$ we are on the $xz$-plane
Points and Vectors in $\mathbb{R}^3$:

Let $v_1, v_2, v_3$ be real numbers.

- A triple $(v_1, v_2, v_3)$ refers to a point in $\mathbb{R}^3$.

- A triple $\langle v_1, v_2, v_3 \rangle$ refers to a vector in $\mathbb{R}^3$, which has tail at $(0,0,0)$ and its head at $(v_1, v_2, v_3)$. 

The origin $(0,0,0)$.

The point $(v_1, v_2, v_3)$.

The vector $\langle v_1, v_2, v_3 \rangle$. 

The diagram illustrates the relationships between points and vectors in three-dimensional space.
Note:

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<th>point</th>
<th>vector</th>
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Example 1. Plot the point \( p(1, 3, -2) \) and the vector \( V = \langle 1, 3, -2 \rangle \).
The length of a vector in \( \mathbb{R}^3 \):

The length of the vector \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) is defined by

\[
|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.
\]

Example 2

Let \( \mathbf{v} = \langle 1, 3, -2 \rangle \), then

\[
|\mathbf{v}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{1 + 9 + 4} = \sqrt{14}.
\]

Vector Operations in \( \mathbb{R}^3 \):

Let \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( c \) be a real number.
<table>
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<th>Operation</th>
<th>Geometrically</th>
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<td>( u + v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle )</td>
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<tr>
<td>( u - v = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle )</td>
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<tr>
<td>( cu = \langle cu_1, cu_2, cu_3 \rangle )</td>
<td>Note: Negative c means you should change direction</td>
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**Example 3**

Let \( u = \langle 1, 3, -2 \rangle \) and \( v = \langle 4, -1, 0 \rangle \). Find

a) \( \frac{1}{2} v \) \( \frac{3}{2} \) \( \frac{4}{2}, \frac{-1}{2}, \frac{-2}{2} = \langle 2, \frac{1}{2}, -1 \rangle \)

b) \( 2u + v = 2 \langle 1, 3, -2 \rangle + \langle 4, -1, 0 \rangle = \langle 2, 6, -4 \rangle + \langle 4, -1, 0 \rangle \)

c) \( u - \frac{1}{4} v \) \( \Rightarrow \) \( \langle 1, 3, -2 \rangle - \frac{1}{4} \langle 4, -1, 0 \rangle = \langle 1, 3, -2 \rangle - \langle 1, \frac{-1}{4}, 0 \rangle = \langle 0, \frac{13}{4}, -2 \rangle \)
Definition of "Dot product"

Given two vectors \( u = \langle u_1, u_2, u_3 \rangle \) and \( v = \langle v_1, v_2, v_3 \rangle \).

The dot product defined by

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.
\]

Note:

There is another form of the dot product:

\[
\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta,
\]

where \( \theta \) is the angle between \( u \) and \( v \) with \( 0 \leq \theta \leq \pi \).
we use the following form of the dot product:

\[ u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 \]

**Example 4:** Compute

\[ \langle 1, 3, -2 \rangle \cdot \langle 4, -1, 0 \rangle = 1(4) + 3(-1) + (-2)0 \]

\[ = 4 - 3 + 0 = 1 \]

**Example 5:** Let \( u = \langle \sqrt{2}, -1, \frac{1}{2} \rangle \) and \( v = \langle -2, 0, \frac{1}{3} \rangle \).

Compute \( u \cdot v \)!!!

\[ u \cdot v = \langle \sqrt{2}, -1, \frac{1}{2} \rangle \cdot \langle -2, 0, \frac{1}{3} \rangle = (-2\sqrt{2}) + (0) + \frac{1}{2} \cdot \frac{1}{3} \]

\[ = -2\sqrt{2} + 0 + \frac{1}{6} = \frac{1}{6} - 2\sqrt{2} \]
Orthogonal vectors:

Two vectors \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) are orthogonal if \( u \cdot v = 0 \). This means that

\[
\sum u_i v_i = 0
\]

If

\[
u_1 v_1 + u_2 v_2 + u_3 v_3 = 0
\]

or

\[
u \cdot v = 0
\]

"\( u \) and \( v \) are orthogonal"

Note:

As an explanation, you can think that if \( u \) and \( v \) are orthogonal, the angle between them is \( \frac{\pi}{2} \).

Then, by the definition \( u \cdot v = |u| |v| \cos \theta \), and the fact that \( \cos \frac{\pi}{2} = 0 \), we get that

\[
u \cdot v = |u| |v| \cos \frac{\pi}{2} = |u| |v| \cdot 0 = 0
\]
Example 6:

a) The vectors \( u = \langle 1, -2, 3 \rangle \) and \( v = \langle -1, 1, 1 \rangle \) are orthogonal. Because
\[
u \cdot v = \langle 1, -2, 3 \rangle \cdot \langle -1, 1, 1 \rangle = (-1) + (-2) + 3 = 0.
\]

b) The vectors \( u = \langle \sqrt{2}, -\frac{1}{2}, 3 \rangle \) and \( v = \langle 0, 4, \frac{2}{3} \rangle \) are orthogonal. Because
\[
u \cdot v = \langle \sqrt{2}, -\frac{1}{2}, 3 \rangle \cdot \langle 0, 4, \frac{2}{3} \rangle = 0 + (-2) + 2 = 0.
\]

Parallel vectors:

Vectors \( u \) and \( v \) are said to be parallel if there is a real number \( c \) such that
\[
u = c \cdot v.
\]
example 7:

a) The vectors \( \mathbf{u} = \langle 1, 2, -1 \rangle \) and \( \mathbf{v} = \langle 2, 4, -2 \rangle \) are parallel.

\[ \langle 1, 2, -1 \rangle = \mathbf{u} = \frac{1}{2} \mathbf{v} = \langle 1, 2, -1 \rangle \]

b) The vectors \( \mathbf{u} = \langle 1, 3, -2 \rangle \) and \( \mathbf{v} = \langle 2, 5, -7 \rangle \) are not parallel.

You cannot find any \( c \in \mathbb{R} \) such that \( \mathbf{u} = c \mathbf{v} \).
In other words, two vectors \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) are Parallel if

\[
\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3}
\]

**Example 8:** Determine which pair of vectors are parallel, orthogonal or neither.

a) \( \langle 2, -1, 5 \rangle \) and \( \langle 1, -3, -1 \rangle \)

b) \( \langle 1, -\frac{1}{2}, \frac{1}{3} \rangle \) and \( \langle -2, 1, \frac{-2}{3} \rangle \)

c) \( \langle 1, -1, 1 \rangle \) and \( \langle -1, -1, 1 \rangle \).

**Solution**

a) \( \langle 2, -1, 5 \rangle \cdot \langle 1, -3, -1 \rangle = 2 + 3 - 5 = 0 \)

So, \( \mathbf{u} \cdot \mathbf{v} = 0 \) and they are orthogonal.

[we don’t need to check if they are parallel, when]

[they are orthogonal]
b). \[ \langle 1, \frac{-1}{2}, \frac{1}{3} \rangle \cdot \langle -2, 1, \frac{-2}{3} \rangle = -2 - \frac{1}{2} - \frac{2}{3} \]
\[= \frac{-12 - 3 - 4}{6} = \frac{-19}{6} \neq 0 \implies \text{not orthogonal} \]

\[ \frac{u_1}{v_1} = \frac{1}{-2}, \quad \frac{u_2}{v_2} = \frac{-1}{2}, \quad \frac{u_3}{v_3} = \frac{1}{\frac{2}{3}} = \frac{-1}{2} \]

So,
\[ \frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = \frac{-1}{2} \implies \text{parallel} \]

C). \[ \langle 1, -1, 2 \rangle \cdot \langle -1, -1, 1 \rangle = -1 + 1 + 1 = 1 \neq 0 \implies \text{not orthogonal} \]

\[ \frac{u_1}{v_1} = \frac{-1}{-1} = 1, \quad \frac{u_2}{v_2} = \frac{-1}{1} = -1, \quad \frac{u_3}{v_3} = \frac{1}{1} = 1 \]

So,
\[ -1 = \frac{u_1}{v_1} \neq \frac{u_2}{v_2} = \frac{u_3}{v_3} = +1 \implies \text{not parallel} \]

The answer would be neither.

Properties of Dot Products: Let \( u, v \) and \( w \) be vectors, and \( c \) be a real number.

1. \( u \cdot v = v \cdot u \) Commutative
2. \( c(u \cdot v) = (cu) \cdot v = u \cdot (cv) \) Associative
3. \( u \cdot (v + w) = u \cdot v + u \cdot w \) Distributive
Example 9: Let $u = \langle 1, 3, -2 \rangle$, $v = \langle 0, -1, 4 \rangle$ and $w = \langle -2, 0, 1 \rangle$. Compute associative

a) $(-3u) \cdot v = -3 (u \cdot v) = (-3)(-11) = 33$.

$u \cdot v = \langle 1, 3, -2 \rangle \cdot \langle 0, -1, 4 \rangle = 0 - 3 - 8 = -11$

b) $u \cdot v + u \cdot w = u \cdot (v + w) = \langle 1, 3, -2 \rangle \cdot \langle -2, -1, 5 \rangle$

$= (-2) + (-3) + (-10) = -15$

$v + w = \langle 0, -1, 4 \rangle + \langle -2, 0, 1 \rangle = \langle -2, -1, 5 \rangle$

Example 10: Find all the values of $a$ such that the vector $\langle a, -3, 2 \rangle$ is orthogonal to the vector $\langle a, 1, -a \rangle$.

We need the condition $\langle a, -3, 2 \rangle \cdot \langle a, 1, -a \rangle = 0$.

On the other hand, $\langle a, -3, 2 \rangle \cdot \langle a, 1, -a \rangle = a^2 - 3 - 2a$.

So, we should solve the equation $a^2 - 3 - 2a = 0$. Its quadratic function $a = \frac{2 \pm \sqrt{4 - (-12)}}{2} = \frac{2 \pm 4}{2} = \{3, -1\}$. 

Dot product of two vectors $= 0$. 