A strongly aperiodic SFT in the Grigorchuk group.

Sebastián Barbieri Lemp

University of British Columbia

Algorithmic questions in dynamical systems
Toulouse
April, 2018
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$. 

Graph showing the relations between $a$, $b$, $c$, $d$, and $id$. The edges indicate the action of the elements on the binary sequences.
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

$x = \begin{array}{cccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\downarrow & b
\end{array} \ldots$

$b(x) =$

\ldots
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

$x = 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ \ldots$

$\downarrow$

$b(x) = 1 \ \ldots$
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

\[ x = 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ldots \]
\[ \downarrow \]
\[ b \ c \ d \]
\[ b(x) = 1 \ 1 \ldots \]
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

\[ x = \begin{array}{cccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\end{array} \ldots \]

\[ \downarrow \]

\[ b \quad c \quad d \quad b \]

\[ b(x) = \begin{array}{cccc}
1 & 1 & 1 & \ldots \\
\end{array} \]
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

\[
x = \begin{array}{cccccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\downarrow \\
b & c & d & b & \ a
\end{array}
\]

\[
b(x) = \begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 0 & \ldots 
\end{array}
\]
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

$x = \begin{array}{ccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ \downarrow & b & c & d & b & a & id \\ b(x) = \begin{array}{ccccccc} 1 & 1 & 1 & 0 & 0 & \end{array} \ldots \end{array}$
The Grigorchuk group

Generated by $a, b, c, d$ acting over $\{0, 1\}^\mathbb{N}$.

$x = 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ldots$

$\downarrow$

$b \ c \ d \ b \ a \ id \ id$

$b(x) = 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ldots$
What about the Grigorchuk group?

- $a, b, c, d$ are involutions.
- Infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word (and conjugacy) problem.
- It has intermediate growth.
- Amenable but not elementary amenable.
- It is commensurable to its square. ie: $G$ and $G \times G$ have an isomorphic finite index subgroup.

The goal of this talk is to construct a strongly aperiodic SFT here.
What about the Grigorchuk group?

- $a, b, c, d$ are involutions.
- Infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word (and conjugacy) problem.
- It has intermediate growth.
- Amenable but not elementary amenable.
- It is commensurable to its square. i.e: $G$ and $G \times G$ have an isomorphic finite index subgroup.

The goal of this talk is to construct a strongly aperiodic SFT here.
Definitions

- $G$ is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $G \curvearrowright \mathcal{A}^G$ is the left shift action given by:

$$ (gx)(h) := x(g^{-1}h). $$
Definitions

- $G$ is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $G \curvearrowleft \mathcal{A}^G$ is the left shift action given by:

$$ (gx)(h) := x(g^{-1}h). $$

Definition: subshift

A closed and shift-invariant set $X \subset \mathcal{A}^G$ is called a subshift.
Definitions

- $G$ is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $G \actson \mathcal{A}^G$ is the left shift action given by:

  $$(gx)(h) := x(g^{-1}h).$$

Definition: subshift

A closed and shift-invariant set $X \subset \mathcal{A}^G$ is called a subshift.

A subshift is a set of configurations avoiding patterns from a list $\mathcal{F}$.

$$p \in \mathcal{A}^S, \quad \llbracket p \rrbracket = \{x \in \mathcal{A}^G \mid x|_S = p\}$$

$$X = X_{\mathcal{F}} = \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} g(\llbracket p \rrbracket)$$
A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_F$ for some finite $F$.
- a *sofic subshift* if $X$ is the image of an SFT by a topological factor (a local recoding).
- an *effectively closed subshift* if $X$ can be defined by a recursively enumerable coding of a set of forbidden patterns.

Strongly aperiodic

A subshift $X \subset \mathcal{A}^G$ is strongly aperiodic if the shift action is free. That is, for all $x \in X$, $gx = x \Rightarrow g = 1$. 

\[\forall x \in X, \quad gx = x \Rightarrow g = 1 \]
Definitions

Classes of subshifts

A subshift $X \subseteq A^G$ is called:

- a **subshift of finite type (SFT)** if $X = X_F$ for some finite $F$.
- a **sofic subshift** if $X$ is the image of an SFT by a topological factor (a local recoding).
Definitions

Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_F$ for some finite $F$.
- a *sofic subshift* if $X$ is the image of an SFT by a topological factor (a local recoding).
- an *effectively closed subshift* if $X$ can be defined by a recursively enumerable coding of a set of forbidden patterns.

Strongly aperiodic

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free.

$$\forall x \in X, \quad gx = x = \Rightarrow g = 1_G.$$
Classes of subshifts

A subshift $X \subset A^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$.
- a *sofic subshift* if $X$ is the image of an SFT by a topological factor (a local recoding).
- an *effectively closed subshift* if $X$ can be defined by a recursively enumerable coding of a set of forbidden patterns.

Strongly aperiodic

A subshift $X \subset A^G$ is *strongly aperiodic* if the shift action is free.

$$\forall x \in X, gx = x \implies g = 1_G.$$

Problem

Question

Which groups admit strongly aperiodic SFTs?
Which groups admit strongly aperiodic SFTs?

Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$
Which groups admit strongly aperiodic SFTs?

Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$
Which groups admit strongly aperiodic SFTs?

Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$
Abelian case

**Proposition**

Every non-empty \(\mathbb{Z}\)-SFT contains a periodic configuration.
Abelian case

**Proposition**
Every non-empty $\mathbb{Z}$-SFT contains a periodic configuration.

There exist strongly aperiodic SFTs on $\mathbb{Z}^2$. 
what’s known? are there any SA SFTs?

result: **nay!**
what’s known? are there any SA SFTs?

result: **nay!**

- (Jeandel ‘15) If $G$ is recursively presented and has undecidable word problem.
what’s known? are there any SA SFTs?

<table>
<thead>
<tr>
<th>Result: nay!</th>
</tr>
</thead>
<tbody>
<tr>
<td>- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.</td>
</tr>
<tr>
<td>- (Cohen ’15) If $G$ has two or more ends.</td>
</tr>
</tbody>
</table>
what’s known? are there any SA SFTs?

result: **nay!**
- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.
- (Cohen ’15) If $G$ has two or more ends.

result: **aye!**
- (Folklore) $\mathbb{Z}^d$ for $d > 1$. 
what’s known? are there any SA SFTs?

**result: nay!**

- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.
- (Cohen ’15) If $G$ has two or more ends.

**result: aye!**

- (Folklore) $\mathbb{Z}^d$ for $d > 1$.
- (Şahin, Schraudner, Ugarcovici, ’$+\infty$ ’14]) The discrete Heisenberg group.
what’s known? are there any SA SFTs?

result: **nay!**
- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.
- (Cohen ’15) If $G$ has two or more ends.

result: **aye!**
- (Folklore) $\mathbb{Z}^d$ for $d > 1$.
- (Şahin, Schraudner, Ugarcovici, ’$+\infty$ ’14) The discrete Heisenberg group.
- (Cohen, Goodman-Strauss, ’15) Surface groups.
what’s known? are there any SA SFTs?

result: **nay!**

- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.
- (Cohen ’15) If $G$ has two or more ends.

result: **aye!**

- (Folklore) $\mathbb{Z}^d$ for $d > 1$.
- (Şahin, Schraudner, Ugarcovici, ’+∞ [’14]) The discrete Heisenberg group.
- (Cohen, Goodman-Strauss, ’15) Surface groups.
- (Cohen, Goodman-Strauss, Rieck, ’17) One-ended Gromov-hyperbolic groups.
what’s known? are there any SA SFTs?

**result: nay!**

- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.
- (Cohen ’15) If $G$ has two or more ends.

**result: aye!**

- (Folklore) $\mathbb{Z}^d$ for $d > 1$.
- (Şahin, Schraudner, Ugarcovici, ’+$\infty$ [’14]) The discrete Heisenberg group.
- (Cohen, Goodman-Strauss, ’15) Surface groups.
- (Cohen, Goodman-Strauss, Rieck, ’17) One-ended Gromov-hyperbolic groups.
- (B, Sablik, ’18+ [’16]) Groups of the form $\mathbb{Z}^d \rtimes_\varphi G$ with $d > 1$, $G$ f.g. and decidable word problem.
what’s known? are there any SA SFTs?

**result: nay!**

- (Jeandel ’15) If $G$ is recursively presented and has undecidable word problem.
- (Cohen ’15) If $G$ has two or more ends.

**result: aye!**

- (Folklore) $\mathbb{Z}^d$ for $d > 1$.
- (Şahin, Schraudner, Ugarcovici, ’$+\infty$ [’14]) The discrete Heisenberg group.
- (Cohen, Goodman-Strauss, ’15) Surface groups.
- (Cohen, Goodman-Strauss, Rieck, ’17) One-ended Gromov-hyperbolic groups.
- (B, Sablik, ’18+ [’16]) Groups of the form $\mathbb{Z}^d \rtimes \varphi G$ with $d > 1$, $G$ f.g. and decidable word problem.
- (Jeandel, ’16) f.g. polycyclic groups which are not virtually $\mathbb{Z}$. 
What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.

∃ SA SFTs in the Grigorchuk group
What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.

\[ \exists \text{SA SFTs in } G_1 \times G_2 \times G_3 \]

\[ \exists \text{SA SFTs in the Grigorchuk group} \]
What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.

Every EC $G$-subshift is a sub-action of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group
What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.

Every EC $G \curvearrowright \{0,1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

Every EC $G$-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group
What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.

Every EC $G \curvearrowright \{0, 1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group

AS 2010, DRS 2010

ABT 2018+ (2015)
We say that two groups $G_1, G_2$ are \textit{commensurable} if they contain finite index subgroups $H_1, H_2$ such that $H_1 \cong H_2$. 

\[ G_1 \leftrightarrow H_1 \cong H_2 \leftrightarrow G_2 \]
We say that two groups $G_1, G_2$ are \textit{commensurable} if they contain finite index subgroups $H_1, H_2$ such that $H_1 \cong H_2$.

\[
G_1 \leftrightarrow H_1 \cong H_2 \leftrightarrow G_2
\]

- Recall that the Grigorchuk group $G$ is commensurable to its square $G \times G$
Commensurability

We say that two groups $G_1, G_2$ are *commensurable* if they contain finite index subgroups $H_1, H_2$ such that $H_1 \cong H_2$.

\[
G_1 \leftrightarrow H_1 \cong H_2 \rightarrow G_2
\]

▷ Recall that the Grigorchuk group $G$ is commensurable to its square $G \times G$
▷ if $G$ is commensurable to $G \times G$, then it is also commensurable to $G \times G \times G$. 

Theorem (Carroll-Penland, 2015)

Admitting a strongly aperiodic SFT is a commensurability invariant.
We say that two groups $G_1$, $G_2$ are *commensurable* if they contain finite index subgroups $H_1$, $H_2$ such that $H_1 \cong H_2$.

\[
G_1 \leftrightarrow H_1 \cong H_2 \rightarrow G_2
\]

Recall that the Grigorchuk group $G$ is commensurable to its square $G \times G$

if $G$ is commensurable to $G \times G$, then it is also commensurable to $G \times G \times G$.

Theorem (Carroll-Penland, 2015)

*Admitting a strongly aperiodic SFT is a commensurability invariant.*
\[ \exists \text{ SA SFTs in } G_1 \times G_2 \times G_3 \]

\[ \exists \text{ SA SFTs in the Grigorchuk group} \]
∃ SA SFTs in $G_1 \times G_2 \times G_3$

\[\xrightarrow{\text{In fact, the same result can be extended to branch groups.}}\]

∃ SA SFTs in the Grigorchuk group
∃ SA SFTs in $G_1 \times G_2 \times G_3$

∃ SA SFTs in branch groups

In fact, the same result can be extended to **branch groups**.
In fact, the same result can be extended to **branch groups**.

**Theorem**

*Let $G$ be a finitely generated and recursively presented branch group. Then $G$ has decidable word problem if and only if there exists a non-empty strongly aperiodic $G$-SFT.*
We want to show next:

Every EC $G$-subshift is a sub-action of a $G \times H_1 \times H_2$-sofic

\[ \exists \text{ SA SFTs in } G_1 \times G_2 \times G_3 \]

\[ \exists \text{ SA EC subshifts in } G \]
We want to show next:

Every EC $G$-subshift is a sub-action of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$
Square-free vertex coloring

Let $G = (V, E)$ be a graph. A vertex coloring is a function $x : V \rightarrow \mathcal{A}$. We say it is square-free if for every odd-length path $p = v_1 \ldots v_{2n}$ then there exists $1 \leq j \leq n$ such that $x(v_j) \neq x(v_{j+n})$. 

$C_5$ has a square-free vertex coloring with 4 colors, but not with 3.
Let $G = (V, E)$ be a graph. A vertex coloring is a function $x : V \to A$. We say it is square-free if for every odd-length path $p = v_1 \ldots v_{2n}$ then there exists $1 \leq j \leq n$ such that $x(v_j) \neq x(v_{j+n})$.

$C_5$ has a square-free vertex coloring with 4 colors, but not with 3.
Some infinite graphs do not admit square-free vertex colorings: $K_{\mathbb{N}}$. 
Square-free vertex coloring

Some infinite graphs do not admit square-free vertex colorings: $K_\mathbb{N}$.

Theorem: Alon, Grytczuk, Haluszczak and Riordan
Every finite graph with maximum degree $\Delta$ can be square-free vertex colored with $2^{17} \Delta^2$ colors.
Some infinite graphs do not admit square-free vertex colorings: $K_\mathbb{N}$.

**Theorem: Alon, Grytczuk, Haluszczak and Riordan**

Every finite graph with maximum degree $\Delta$ can be square-free vertex colored with $2^{17}\Delta^2$ colors.

Let

$$\Gamma(G, S) = (G, \{\{g, gs\}, g \in G, s \in S\})$$

be the undirected right Cayley graph of $G$ with respect to $S \subseteq G$. A compactness argument shows:

**Theorem**

$\Gamma(G, S)$ can be square-free vertex colored with $2^{19}|S|^2$ colors.
Let $|A| \geq 2^{19}|S|^2$ and $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.
Constructing an EC SA subshift

Let $|A| \geq 2^{19}|S|^2$ and $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$. 
Constructing an EC SA subshift

Let $|A| \geq 2^{19}|S|^2$ and $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.
- Let $g \in G$ such that $gx = x$ for some $x \in X$. 

If $G$ has decidable word problem, then $X$ is effectively closed.
Constructing an EC SA subshift

Let $|A| \geq 2^{19}|S|^2$ and $X \subset \mathcal{A}^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.
- Let $g \in G$ such that $gx = x$ for some $x \in X$.
- Factorize $g$ as $uvw$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$. 

If $G$ has decidable word problem, then $X$ is effectively closed.
Constructing an EC SA subshift

Let $|A| \geq 2^{19}|S|^2$ and $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.
- Let $g \in G$ such that $gx = x$ for some $x \in X$.
- Factorize $g$ as $uvw$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \ldots w_n$ and consider the odd length walk $\pi = v_0 v_1 \ldots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$$v_i = \begin{cases} 
1_G & \text{if } i = 0 \\
 w_1 \ldots w_i & \text{if } i \in \{1, \ldots, n\} \\
 ww_1 \ldots w_{i-n} & \text{if } i \in \{n+1, \ldots, 2n-1\}
\end{cases}$$
Constructing an EC SA subshift

Let $|\mathcal{A}| \geq 2^{19}|S|^2$ and $X \subset \mathcal{A}^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.
- Let $g \in G$ such that $gx = x$ for some $x \in X$.
- Factorize $g$ as $uvw$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \ldots w_n$ and consider the odd length walk $\pi = v_0 v_1 \ldots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$\begin{align*}
v_i &= \begin{cases} 
1_G & \text{if } i = 0 \\
 w_1 \ldots w_i & \text{if } i \in \{1, \ldots, n\} \\
ww_1 \ldots w_{i-n} & \text{if } i \in \{n+1, \ldots, 2n-1\} 
\end{cases}
\end{align*}$

- $\pi$ is a path and $x_{v_i} = x_{v_{i+n}}$. $\rightarrow \leftarrow$
Let $|\mathcal{A}| \geq 2^{19}|S|^2$ and $X \subset \mathcal{A}^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.

- Let $g \in G$ such that $gx = x$ for some $x \in X$.

- Factorize $g$ as $uvw$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.

- If not, let $w = w_1 \ldots w_n$ and consider the odd length walk $\pi = v_0 v_1 \ldots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

  \[ v_i = \begin{cases} 
  1_G & \text{if } i = 0 \\
  w_1 \ldots w_i & \text{if } i \in \{1, \ldots, n\} \\
  ww_1 \ldots w_{i-n} & \text{if } i \in \{n + 1, \ldots, 2n - 1\}
  \end{cases} \]

- $\pi$ is a path and $x_{v_i} = x_{v_{i+n}}$.  

- Therefore, $g = 1_G$. 

If $G$ has decidable word problem, then $X$ is effectively closed.
Constructing an EC SA subshift

Let $|A| \geq 2^{19}|S|^2$ and $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.
- Let $g \in G$ such that $gx = x$ for some $x \in X$.
- Factorize $g$ as $uwv$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \ldots w_n$ and consider the odd length walk $\pi = v_0v_1 \ldots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \ldots w_i & \text{if } i \in \{1, \ldots, n\} \\ w w_1 \ldots w_{i-n} & \text{if } i \in \{n+1, \ldots, 2n-1\} \end{cases}$$

- $\pi$ is a path and $x_{v_i} = x_{v_{i+n}}$. $\rightarrow \leftarrow$
- Therefore, $g = 1_G$.

If $G$ has decidable word problem, then $X$ is effectively closed.
We want to show next:

Every EC $G$-subshift is a sub-action of a $G \times H_1 \times H_2$-sofic

\[ \exists \text{ SA SFTs in } G_1 \times G_2 \times G_3 \]

\[ \exists \text{ SA EC subshifts in } G \]
We want to show next:

Every EC $G$-subshift is a sub-action of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$
The philosophy behind it

Finitely presented group

A group $G$ is finitely presented if $G \cong \langle S \mid R \rangle$ where both $S$ and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$
The philosophy behind it

**Finitely presented group**

A group $G$ is finitely presented if $G \cong \langle S | R \rangle$ where both $S$ and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1} \rangle$$

**Recursively presented group**

A group $G$ is recursively presented if $G \cong \langle S | R \rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursively enumerable sets.

$$L = \langle a, t | (at^n at^{-n})^2, n \in \mathbb{N} \rangle$$
The philosophy behind it

Theorem (Higman, 1961)

For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$. 

Corollary (Theorem: Novikov 1955, Boone 1958)

There are finitely presented groups with undecidable word problem.

Just apply Higman's theorem to $G = \langle a, b, c, d \mid b^n a b^n = c^n d, n \in \text{HALT} \rangle$... done!
The philosophy behind it

Theorem (Higman, 1961)

For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$.

“A complicated object is realized inside another object which admits a much simpler presentation.”
The philosophy behind it

**Theorem (Higman, 1961)**

For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$.

“A complicated object is realized inside another object which admits a much simpler presentation.”

**Corollary [Theorem: Novikov 1955, Boone 1958]**

There are finitely presented groups with undecidable word problem

Just apply Higman’s theorem to

$G = \langle a, b, c, d \mid b^{-n}ab^n = c^{-n}dc^n, n \in \text{HALT} \rangle$... done!
The case of subshifts

Every EC $\mathbb{Z}$-subshift $X$ is a subaction of a $\mathbb{Z}^2$-sofic $Y$
The case of subshifts
The case of subshifts
In our case

proof

- Take $G_1$ EC SA subshift. Use simulation to obtain a $G_1 \times G_2 \times G_3$-sofic subshift $Y_1$ such that $G_2 \times G_3$ act trivially and $G_1$ acts freely.
- Do the same for $G_2, G_3$ to get $Y_2, Y_3$.
- $Y_1 \times Y_2 \times Y_3$ is a SA sofic subshift.
- Any SFT extension $X \rightarrow Y_1 \times Y_2 \times Y_3$ works.
In our case

proof

- Take $G_1$ EC SA subshift. Use simulation to obtain a $G_1 \times G_2 \times G_3$-sofic subshift $Y_1$ such that $G_2 \times G_3$ act trivially and $G_1$ acts freely.
- Do the same for $G_2, G_3$ to get $Y_2, Y_3$.
- $Y_1 \times Y_2 \times Y_3$ is a SA sofic subshift.
- Any SFT extension $X \hookrightarrow Y_1 \times Y_2 \times Y_3$ works.

Every EC $G$-subshift is a sub-action of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$
How does one prove such a thing?

Every EC $G \curvearrowright \{0, 1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $G$-subshift is a subaction of a $G \times H_1 \times H_2$-sofic
How does one prove such a thing?

Every EC $G \curvearrowright \{0,1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $G$-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

Two ingredients:
- A Toeplitz coding of EC actions from a work of me and M. Sablik.
- A coding of E. Jeandel of a theorem of Seward on translation-like actions.
How does one prove such a thing?

Let’s keep it simple, let’s do $G \times \mathbb{Z}^2$. Consider an action

$$G \curvearrowright X \subset \{0, 1\}^\mathbb{N}$$
How does one prove such a thing?

Let’s keep it simple, let’s do \( G \times \mathbb{Z}^2 \). Consider an action

\[
G \curvearrowright X \subset \{0, 1\}^\mathbb{N}
\]

Let \( \Psi : \{0, 1\}^\mathbb{N} \rightarrow \{0, 1, \$\}^\mathbb{Z} \) be given by:

\[
\Psi(x)_j = \begin{cases} 
  x_n & \text{if } j = 3^n \mod 3^{n+1} \\
  \$ & \text{in the contrary case.}
\end{cases}
\]
How does one prove such a thing?

Let’s keep it simple, let’s do \( G \times \mathbb{Z}^2 \). Consider an action

\[
G \ltimes X \subset \{0, 1\}^N \]

Let \( \Psi : \{0, 1\}^N \to \{0, 1, \$\}^\mathbb{Z} \) be given by:

\[
\Psi(x)_j = \begin{cases} 
  x_n & \text{if } j = 3^n \mod 3^{n+1} \\
  $ & \text{in the contrary case.}
\end{cases}
\]

**Example**

If we write \( x = x_0x_1x_2x_3 \ldots \) we obtain,

\[
\Psi(x) = \ldots x_0x_1x_0\$x_2x_0x_1x_0\$x_0\$x_0x_1x_0\$x_0\$x_3x_0 \ldots
\]
How does one prove such a thing?

\[ \ldots \cdot x_0 \cdot x_1 \cdot x_0 \cdot x_0 \cdot x_1 \cdot x_0 \cdot x_0 \cdot x_2 \cdot x_0 \cdot x_1 \cdot x_0 \cdot x_0 \cdot x_1 \cdot x_0 \cdot x_0 \cdot x_3 \cdot x_0 \cdot \ldots \]
How does one prove such a thing?

\[
\ldots x_0 x_1 x_0 \ldots x_0 x_2 x_0 x_1 x_0 \ldots x_0 x_1 x_0 \ldots x_0 x_3 x_0 \ldots
\]

↓

\[
\ldots x_0 x_1 x_0 \ldots x_0 x_2 x_0 x_1 x_0 \ldots x_0 x_2 x_0 x_3 x_0 \ldots
\]
How does one prove such a thing?

\[ \ldots x_0 x_1 x_0 x_2 x_0 x_1 x_0 x_0 x_1 x_0 x_0 x_1 x_0 x_0 x_3 x_0 \ldots \]

\[ \downarrow \]

\[ \ldots x_0 x_1 x_0 x_2 x_0 x_1 x_0 x_0 x_1 x_0 x_0 x_1 x_0 x_0 x_3 x_0 \ldots \]

\[ \downarrow \]

\[ \ldots x_1 x_2 x_1 x_1 x_1 x_1 x_1 x_3 \ldots \]
How does one prove such a thing?

\[ \ldots x_0 x_1 x_0 x_0 x_2 x_0 x_1 x_0 x_0 x_0 x_0 x_1 x_0 x_0 x_3 x_0 \ldots \]

\[ \downarrow \]

\[ \ldots x_0 x_1 x_0 x_0 x_2 x_0 x_1 x_0 x_0 x_0 x_0 x_1 x_0 x_0 x_0 x_0 x_3 x_0 \ldots \]

\[ \downarrow \]

\[ \ldots x_1 x_2 x_1 x_1 x_1 x_3 x_3 x_2 x_1 x_1 x_1 x_1 x_3 x_3 x_4 x_1 \ldots \]
How does one prove such a thing?

- pick a finite set of generators \( S \) of \( G \).
- construct a subshift \( \Pi \) where every configuration is (up to small details) an \( S \)-tuple of configurations of the previous form.

\[
S = \{1_G, s_1, \ldots, s_n\}
\]

\[
\begin{pmatrix}
\Psi(x) \\
\Psi(s_1(x)) \\
\vdots \\
\Psi(s_n(x))
\end{pmatrix} \in \Pi
\]
How does one prove such a thing?

▷ pick a finite set of generators $S$ of $G$.
▷ construct a subshift $\Pi$ where every configuration is (up to small details) an $S$-tuple of configurations of the previous form.

$$S = \{1_G, s_1, \ldots s_n\}$$

$$
\begin{pmatrix}
\Psi(x) \\
\Psi(s_1(x)) \\
\vdots \\
\Psi(s_n(x))
\end{pmatrix} \in \Pi
$$

Claim

If $G \curvearrowright X$ is an effectively closed action, $\Pi$ is an effectively closed subshift.
How does one prove such a thing?

Every EC $G \curvearrowright \{0,1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

▷ There exists a sofic $\mathbb{Z}^2$-subshift $\tilde{\Pi}$ having $\Pi$ in every horizontal row.
How does one prove such a thing?

Every EC $G \curvearrowright \{0, 1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

There exists a sofic $\mathbb{Z}^2$-subshift $\tilde{\Pi}$ having $\Pi$ in every horizontal row.

Using the decoding argument, construct a map from $\tilde{\Pi}$ to $X$. 

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic
How does one prove such a thing?

Every EC $G \curvearrowright \{0,1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

▷ There exists a sofic $\mathbb{Z}^2$-subshift $\tilde{\Pi}$ having $\Pi$ in every horizontal row.

▷ Using the decoding argument, construct a map from $\tilde{\Pi}$ to $X$.

▷ Put in every $G$-coset of $G \times \mathbb{Z}^2$ a configuration of $\tilde{\Pi}$. Tie them using local rules.
How does one prove such a thing?

\[ \begin{pmatrix} \psi(z) \\ \psi(s_1(z)) \\ \vdots \\ \psi(s_n(z)) \end{pmatrix} \in \tilde{\Pi} \]

\[ \begin{pmatrix} \psi(y) \\ \psi(s_1(y)) \\ \vdots \\ \psi(s_n(y)) \end{pmatrix} \in \tilde{\Pi} \]

\[ \begin{pmatrix} \psi(x) \\ \psi(s_1(x)) \\ \vdots \\ \psi(s_n(x)) \end{pmatrix} \in \tilde{\Pi} \]
How does one prove such a thing?

\[ \begin{pmatrix} \Psi(z) \\ \Psi(s_1(z)) \\ \vdots \\ \Psi(s_n(z)) \end{pmatrix} \in \tilde{\Pi} \]

\[ \begin{pmatrix} \Psi(y) \\ \Psi(s_1(y)) \\ \vdots \\ \Psi(s_n(y)) \end{pmatrix} \in \tilde{\Pi} \]

\[ \begin{pmatrix} \Psi(x) \\ \Psi(s_1(x)) \\ \vdots \\ \Psi(s_n(x)) \end{pmatrix} \in \tilde{\Pi} \]
How does one prove such a thing?

\[
\begin{pmatrix}
\Psi(z) \\
\Psi(s_1(z)) \\
\vdots \\
\Psi(s_n(z))
\end{pmatrix} \in \tilde{\Pi}
\]

\[
\begin{pmatrix}
\Psi(y) \\
\Psi(s_1(y)) \\
\vdots \\
\Psi(s_n(y))
\end{pmatrix} \in \tilde{\Pi}
\]

\[
\begin{pmatrix}
\Psi(x) \\
\Psi(s_1(x)) \\
\vdots \\
\Psi(s_n(x))
\end{pmatrix} \in \tilde{\Pi}
\]
How does one prove such a thing?

\[
\begin{pmatrix}
\psi(s_1 s_1(x)) \\
\psi(s_1 s_2 s_1(x)) \\
\vdots \\
\psi(s_n s_1(x))
\end{pmatrix} \in \tilde{\Pi}
\]

\[
\begin{pmatrix}
\psi(s_1(x)) \\
\psi(s_1 s_1(x)) \\
\vdots \\
\psi(s_n s_1(x))
\end{pmatrix} \in \tilde{\Pi}
\]

\[
\begin{pmatrix}
\psi(x) \\
\psi(s_1(x)) \\
\vdots \\
\psi(s_n(x))
\end{pmatrix} \in \tilde{\Pi}
\]
From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?
From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

[Whyte] translation-like action

An action $G \curvearrowright (X, d)$ is \textit{translation-like} if:

- $G$ acts freely
- For each $g \in G$, $\sup_{x \in X} d(x, gx) < \infty$. 

Theorem (Seward, 2013)

Each infinite and f.g. group admits a translation-like action of $\mathbb{Z}$.

This means that each infinite and f.g. group admits a Cayley graph that can be partitioned into disjoint bi-infinite paths.
From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

[Whyte] translation-like action

An action $G \curvearrowright (X, d)$ is translation-like if:

- $G$ acts freely
- For each $g \in G$, $\sup_{x \in X} d(x, gx) < \infty$.

Theorem (Seward, 2013)

Each infinite and f.g. group admits a translation-like action of $\mathbb{Z}$.
From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

[Whyte] translation-like action

an action $G \curvearrowright (X, d)$ is translation-like if:

- $G$ acts freely
- For each $g \in G$, $\sup_{x \in X} d(x, gx) < \infty$.

Theorem (Seward, 2013)

Each infinite and f.g. group admits a translation-like action of $\mathbb{Z}$.

This means that each infinite and f.g. group admits a Cayley graph that can be partitioned into disjoint bi-infinite paths.
[Jeandel] Use the set of generators of the Cayley graph to define an SFT which codes the translation-like action.

Figure: Finding a grid in $H_1 \times H_2$
Every EC $G \curvearrowleft \{0, 1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $G$-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group
Every EC $G \curvearrowright \{0, 1\}^\mathbb{N}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

Every EC $G$-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in $G$

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group
Thank you for your attention!