The domino problem for structures between $\mathbb{Z}$ and $\mathbb{Z}^2$.

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Consider a group $G$.

- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of functions $x : G \rightarrow \mathcal{A}$.
- $\sigma : G \times \mathcal{A}^G \rightarrow \mathcal{A}^G$ is the shift action given by:
  \[
  \sigma_g(x)_h = x_{g^{-1}h}.
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**Definition:** $G$-subshift

$X \subset \mathcal{A}^G$ is a $G$-subshift if it is invariant under the action of $\sigma$ and closed for the product topology on $\mathcal{A}^G$. 
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**Alternative definition : G-subshift**

$X$ is a $G$-subshift if it can be defined as the set of configurations which avoid a set forbidden patterns: $\exists F \subset \bigcup_{F \subset G, |F| < \infty} \mathcal{A}^F$ such that:

\[
X = X_F := \{ x \in \mathcal{A}^G \mid \forall p \in F : p \not\sqsubseteq x \}.
\]
Example in $\mathbb{Z}^2$ : Fibonacci shift

**Example : Fibonacci shift.** $X_{Fib}$ is the set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.
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Example: one-or-less subshift

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\[ X_{\leq 1} := \{ x \in \{0, 1\}^{\mathbb{Z}^d} \mid |\{ z \in \mathbb{Z}^d : x_z = 1\}| \leq 1 \}. \]
Fibonacci in $F_2$.

$\mathcal{F} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Subshifts of finite type.

What about if we only consider local rules?

**Definition : subshift of finite type.**

A $G$-subshift is of finite type (SFT) if it can be defined by a finite set $\mathcal{F}$ of forbidden patterns.

**Example :** Both Fibonacci subshifts shown before are of finite type. $X_{\leq 1}$ isn’t.
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Given a finite set of forbidden patterns, can we decide if the $G$-subshift produced by them is non-empty?
The domino problem.

▶ Every finite alphabet can be identified as a finite subset of \( \mathbb{N} \).

**Domino problem.**

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\text{DP}(G) = \{ \mathcal{F} \subseteq \mathbb{N}_G^* \mid |\mathcal{F}| < \infty, X_{\mathcal{F}} \neq \emptyset \}. 
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If \( G \) is finitely generated by the set \( S \), we can codify each pattern as a function from a finite set of words in \((S \cup S^{-1})^*\) to \( \mathbb{N} \).

Therefore, \( \text{DP}(G) \) can be written as a formal language. We say \( G \) has decidable domino problem if \( \text{DP}(G) \) is Turing-decidable.
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**Question:** Which groups have decidable domino problem?
The easy case $G = \mathbb{Z}$.

**Theorem:**

The set of configurations of a $\mathbb{Z}$-SFT can be characterized as the set of bi-infinite walks in a finite graph.
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Example:
Consider the Fibonacci shift given by $\mathcal{F} = \{11\}$.
The easy case \( G = \mathbb{Z} \).

**Theorem :**
The set of configurations of a \( \mathbb{Z} \)-SFT can be characterized as the set of bi-infinite walks in a finite graph.

**Example :** Consider the Fibonacci shift given by \( \mathcal{F} = \{11\} \).

As the graph is finite, a \( \mathbb{Z} \)-SFT is non-empty if and only if its Rauzy graph contains a cycle, thus \( \text{DP}(\mathbb{Z}) \) is decidable.
The not so easy case: \( G = \mathbb{Z}^2 \)

The name "Domino problem" comes from the \( G = \mathbb{Z}^2 \) case.
The not so easy case: $G = \mathbb{Z}^2$

The name "Domino problem" comes from the $G = \mathbb{Z}^2$ case. Wang tiles are unit squares with colored edges, the forbidden patterns are implicit in the alphabet.
Wang’s conjecture (1961)

If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.
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If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.

If Wang’s conjecture is true, we can decide if a set of Wang tiles can tile the plane!

Semi-algorithm 1:
1. Accept if there is a periodic configuration.
2. loops otherwise

Semi-algorithm 2:
1. Accept if a block $[0, n]^2$ cannot be tiled without breaking local rules.
2. loops otherwise
Wang’s conjecture

Theorem [Berger 1966]

Wang’s conjecture is FALSE

His construction encodes a Turing machine using an alphabet of size 20426.

His proof was later simplified by Robinson [1971]. A proof with a different approach was also presented by Kari [1996].
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The Robinson tileset, where tiles can be rotated.
General structure of the Robinson tiling

Macro-tiles of level 1.
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Macro-tiles of level 1.

They behave like large □.
From macro-tiles of level 1 to macro-tiles of level 2
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From macro-tiles of level $n$ to macro-tiles of level $n+1$
Some recent results and facts in f.g. groups

- If a group $G$ has undecidable word problem $\Rightarrow \text{DP}(G)$ is undecidable.
- Virtually free groups have decidable domino problem.
- For virtually nilpotent groups: $\text{DP}(G)$ is decidable if and only if it has two or more ends (2013 Ballier, Stein).
- Every virtually polycyclic group which is not virtually $\mathbb{Z}$ has undecidable domino problem (work in progress by Jeandel).
- The domino problem is a quasi-isometry invariant for finitely presented groups (2015 Cohen).
So far we have:

- \( \text{DP}(\mathbb{Z}) \) is decidable.
- \( \text{DP}(\mathbb{Z}^2) \) is undecidable.

And if \( H \leq \mathbb{Z}^2 \), then either \( H \cong 1 \), \( H \cong \mathbb{Z} \) or \( H \cong \mathbb{Z}^2 \).
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And if $H \leq \mathbb{Z}^2$, then either $H \cong 1$, $H \cong \mathbb{Z}$ or $H \cong \mathbb{Z}^2$.

We need to lose the group structure if we want to study intermediate structures.
Toy case: Sierpiński triangle
Coding subsets of $\mathbb{Z}^2$ as configurations.

Let $F \subset \mathbb{Z}^2$ and define the configuration $x_F \in \{0, 1\}^{\mathbb{Z}^2}$:

$$(x_F)_z = \begin{cases} 
1 & \text{if } z \in F \\
0 & \text{if not.}
\end{cases}$$

And let $Y = \bigcup_{z \in \mathbb{Z}^2} \{\sigma_z(x_F)\}$. 

Given a set of forbidden patterns $F$ we can define colorings of $F$ as the configurations of $x_F$ over an alphabet $A \ni 0$ such that the application $\pi : A^{\mathbb{Z}^2} \to \{0, 1\}^{\mathbb{Z}^2}$:

$$\pi(x)_z = \begin{cases} 
1 & \text{if } x_z \neq 0 \\
0 & \text{if } x_z = 0
\end{cases} \text{ yields an element of } Y.$$
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yields an element of $Y$. 
Let \( Y \ni 0^{\mathbb{Z}^2} \) be a \( \mathbb{Z}^2 \)-subshift over the alphabet \( \{0, 1\} \).

Let \( \mathcal{F} \) be a set of forbidden patterns over an alphabet \( \mathcal{A} \ni 0 \) which does not forbid any pattern consisting only of 0.
Formally...

Let $Y \ni 0^\mathbb{Z}^2$ be a $\mathbb{Z}^2$-subshift over the alphabet $\{0, 1\}$.

Let $\mathcal{F}$ be a set of forbidden patterns over an alphabet $\mathcal{A} \ni 0$ which does not forbid any pattern consisting only of 0.

**Definition : $Y$-based subshift**

The $Y$-based subshift defined by $\mathcal{F}$ is the set:

$$X_{Y, \mathcal{F}} := \pi^{-1}(Y) \cap X_{\mathcal{F}}.$$
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**Definition : $Y$-based domino problem**

$$\text{DP}(Y) := \{ \mathcal{F} \subset \mathbb{N}^*_\mathbb{Z}^2 \mid |\mathcal{F}| < \infty \text{ and } X_{Y,\mathcal{F}} \neq \{0^\mathbb{Z}^2\} \}.$$
We focus on subshifts $Y$ with a self-similar structure generated by substitutions.
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- If $Y$ contains a strongly periodic point which is not $0^\mathbb{Z}^2$ then $\text{DP}(Y)$ is undecidable.

- It is easy to calculate a Hausdorff dimension (in this case box-counting dimension). Is there a threshold in the dimension which enforces undecidability?

- These subshifts can be defined by local rules (sofic subshifts) according to Mozes Theorem.
Back to the fractal structures...

We focus on subshifts $Y$ with a self-similar structure generated by substitutions.

- If $Y$ contains a strongly periodic point which is not $0^\mathbb{Z^2}$ then $\text{DP}(Y)$ is undecidable.
- It is easy to calculate a Hausdorff dimension (in this case box-counting dimension). Is there a threshold in the dimension which enforces undecidability?
- These subshifts can be defined by local rules (sofic subshifts) according to Mozes Theorem.

In particular we consider: substitutions over $\{0, 1\}$ such that the image of 0 is a rectangle of zeros.
Consider the alphabet $\mathcal{A} = \{\square, ■\}$ and the self-similar substitution $s$ such that:

\[
\begin{align*}
\square & \rightarrow \begin{array}{c}
\text{a square}
\end{array} \\
■ & \rightarrow \begin{array}{c}
\text{a larger square with a smaller square inside}
\end{array}
\end{align*}
\]

and

\[
\begin{array}{c}
\text{a square with a smaller square inside}
\end{array} \\
\begin{array}{c}
\text{a larger square with a smaller square inside}
\end{array}
\]

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Example 2 : Sierpiński carpet
Example 3: The Bridge.
Theorem:

The domino problem is decidable in the Sierpiński triangle.
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Proof strategy:

- Consider a rectangle containing the union of the support of all forbidden patterns.
Toy case 1: Sierpiński triangle.

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- Suppose we can tile locally an iteration $n$ of the substitution without producing forbidden patterns.
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▶ Consider a rectangle containing the union of the support of all forbidden patterns.

▶ Suppose we can tile locally an iteration $n$ of the substitution without producing forbidden patterns.

▶ To construct a tiling of the next level, it suffices to "paste" three tilings of the iteration $n$ without producing forbidden patterns.
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Proof strategy (continued):

▶ Keep the information about the pasting places (finite tuples) and build pasting rules \((T_1, T_2, T_3) \rightarrow T_4\).
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Proof strategy (continued) :

▶ Keep the information about the pasting places (finite tuples) and build pasting rules \((T_1, T_2, T_3) \rightarrow T_4\).

▶ For each iteration \(n\), construct the set of tuples observed in the pasting places. Construct the next set using this one.
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Proof strategy (continued):

▶ Keep the information about the pasting places (finite tuples) and build pasting rules $(T_1, T_2, T_3) \rightarrow T_4$.

▶ For each iteration $n$, construct the set of tuples observed in the pasting places. Construct the next set using this one.

▶ This process either cycles (arbitrary iterations can be tiled) or ends up producing the empty set (the only valid tiling is $0^{\mathbb{Z}^2}$).
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- This process either cycles (arbitrary iterations can be tiled) or ends up producing the empty set (the only valid tiling is \(0^{\mathbb{Z}^2}\)).

This technique can be extended to a big class of self-similar substitutions!
Theorem:
The domino problem is undecidable in the Sierpiński carpet.

Proof strategy:
▶ Suppose we can simulate substitutions over the Sierpiński carpet (using a bigger alphabet).
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Toy case 2: Sierpiński carpet.

\[
\begin{align*}
\bullet & \rightarrow s' \downarrow \leftrightarrow \uparrow 0 \uparrow \downarrow, \\
\bullet & \leftrightarrow \bullet \\
\bullet & \leftrightarrow \bullet \\
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\leftrightarrow & \rightarrow s' \leftrightarrow \leftrightarrow \leftrightarrow \bullet \leftrightarrow \leftrightarrow \leftrightarrow \bullet.
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- Suppose we can simulate substitutions over the Sierpiński carpet (using a bigger alphabet).
- Use the substitution shown above to simulate arbitrarily big patterns of a $\mathbb{Z}^2$-subshift.
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- $\text{DP}(\mathbb{Z}^2)$ is reduced to the domino problem in the carpet.
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The domino problem is undecidable in the Sierpiński carpet.

Proof strategy:

▶ Suppose we can simulate substitutions over the Sierpiński carpet (using a bigger alphabet).

- Use the substitution shown above to simulate arbitrarily big patterns of a $\mathbb{Z}^2$-subshift
- $\text{DP}(\mathbb{Z}^2)$ is reduced to the domino problem in the carpet.

It only remains to show that we can simulate substitutions with local rules.
We need to prove a modified version of Mozes’ theorem:

**Theorem : Mozes.**

The subshifts generated by $\mathbb{Z}^2$-substitutions are sofic (are the image of an SFT under a cellular automaton)

We can prove a similar version for some $Y$-based subshifts. Among them the Sierpiński carpet.
Toy case 2: Sierpiński carpet and Mozes
Conclusion

We can generalize the ideas in the previous toy problems to attack classes of substitutions:
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Bounded Connectivity

Decidable domino problem
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**Strong grid**

- ![Strong grid example]

**Undecidable domino problem**

- ![Undecidable domino example]
Conclusion

And separate the substitutions which we cannot classify into two groups:
And separate the substitutions which we cannot classify into two groups:

Isthmus

Unknown
Conclusion

And separate the substitutions which we cannot classify into two groups:

Weak grid

Unknown
We got some ideas of how it might be...
We don’t know anything about this one.
And about the Hausdorff dimension?...
And about the Hausdorff dimension?...

There is no threshold.
Thank you for your attention!