5 Symmetries & The Spin

- State in Hilbert space $\{\lambda\psi\}$ where $\|\psi\|=1$ and $|\lambda|=1$ $\lambda\in\mathbb{C}$
Best described by the corresponding projector P_ψ:
$$P_\psi\phi = \langle\psi,\phi\rangle\psi$$
Space of 1-d projectors in $\mathcal{H}$: $P\mathcal{H}$
Def. A symmetry is a map $S: P\mathcal{H}\to P\mathcal{H}$ s.t.
$$\mathrm{Tr}(SP_1 SP_2) = \mathrm{Tr}(P_1P_2)$$
For any $\psi_1 = P_1\psi$, $\psi_2 = P_2\psi_2$ normalized:
$\psi_1' = SP_1\psi_1$; $\psi_2' = SP_2\psi_2$
$$|\langle\psi_2',\psi_1'\rangle|^2 = |\langle\psi_2,\psi_1\rangle|^2$$
namely:
S leaves all quantum mechanical probabilities invariant.
- An operator $A\in\mathcal{L}(\mathcal{H})$ is antilinear if
$$A(\alpha\psi+\phi) = \bar\alpha A\psi + A\phi$$
Note $(\alpha A)^\dagger = A^\dagger\bar\alpha = \alpha A^\dagger$
- Theorem [Wigner] Let S be a symmetry. Then there is $U\in\mathcal{L}(\mathcal{H})$ such that
$$SP = UPU^\dagger$$
where U is a linear or antilinear isometry.
If S is invertible, then U is (anti)unitary.
U is uniquely determined by S up to $z\in\mathbb{C}$, $|z|=1$.

Note. Isometry $\langle U\psi, U\phi\rangle = \begin{cases}\langle\psi,\phi\rangle\\ \langle\phi,\psi\rangle\end{cases}$

and such a map is necessarily linear, resp antilinear (because the inner product is so).
Indeed:
$$\langle U\psi, U(\phi_1+\phi_2)\rangle = \begin{cases}\langle\psi,\phi_1+\phi_2\rangle = \langle\psi\phi_1\rangle+\langle\psi\phi_2\rangle\\ \langle\phi_1+\phi_2,\psi\rangle = \langle\phi_1,\psi\rangle+\langle\phi_2,\psi\rangle\end{cases}$$

$$= \langle U\psi, U\phi_1 \rangle + \langle U\psi, U\phi_2 \rangle \quad \text{in both cases.}$$

$$= \langle U\psi, U\phi_1 + U\phi_2 \rangle$$

Hence: $U(\phi_1 + \phi_2) = U\phi_1 + U\phi_2$ indeed.

Proof of Wigner's theorem.

- Step 1: Extend S from $P\mathcal{H}$ to $C(\mathcal{H})$, the linear hull of $P\mathcal{H}$:

$$C(\mathcal{H}) = \left\{ \sum_{i=1}^{N} \lambda_i P_i : P_i \in P\mathcal{H}, \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}.$$

$$S \sum \lambda_i P_i = \begin{cases} \lambda_i SP_i & \text{(linear extension)} \\ \bar{\lambda}_i SP_i & \text{(antilinear extension)} \end{cases}$$

Claim: For any $A, B \in C(\mathcal{H})$:

$$\mathrm{Tr}((SA)^\dagger (SB)) = \begin{cases} \overline{\mathrm{Tr}(A^\dagger B)} & (L) \\ \mathrm{Tr}(A^\dagger B) & (A) \end{cases} \qquad (1)$$

Indeed: $\mathrm{Tr}((SA)^\dagger (SB)) = \sum_{i,j} \begin{Bmatrix} \bar{\lambda}_i \\ \lambda_i \end{Bmatrix} \cdot \begin{Bmatrix} \mu_j \\ \bar{\mu}_j \end{Bmatrix} \underbrace{\mathrm{Tr}((SP_i)^\dagger S\tilde{P}_j)}$

and since S is a symmetry: $= \mathrm{Tr}(P_i^\dagger P_j)$

$$= \begin{cases} \mathrm{Tr}\left(\left(\sum_i \lambda_i P_i\right)^\dagger \left(\sum_j \mu_j \tilde{P}_j\right) \right) = \mathrm{Tr}(A^\dagger B) \\ \mathrm{Tr}\left(\left(\sum_i \lambda_i P_i\right) \left(\sum_j \mu_j \tilde{P}_j\right)^\dagger \right) = \mathrm{Tr}(AB^\dagger) = \overline{\mathrm{Tr}(A^\dagger B)} \end{cases}$$

In particular, the extension is injective: If $SA = 0$, then $0 = \mathrm{Tr}((SA)^\dagger (SA)) = \mathrm{Tr}(A^\dagger A)$ and hence $A = 0$.

Claim: If $A, B \in C(\mathcal{H})$ are self-adjoint and $[A, B] = 0$,

(*) then $S(A)S(B) = S(AB)$.

Indeed: A, B are simultaneously diagonalizable

$$A = \sum_i \lambda_i(A) P_i \quad ; \quad B = \sum_i \lambda_i(B) P_i$$

with $P_i P_j = \delta_{ij} P_i$. But then

$$\mathrm{Tr}\left((SP_i)^*(SP_j)\right) = \mathrm{Tr}(P_i^* P_j) = \delta_{ij}$$ and hence

$$S(A)S(B) = \sum_{i,j} \lambda_i(A)\lambda_j(B) \underbrace{(SP_i)(SP_j)}_{=0 \text{ if } i\neq j} = \sum_i \underbrace{\lambda_i(A)\lambda_i(B)}_{=\lambda_i(AB)} SP_i$$

$$= S(AB)$$

Finally, we note that $C(\mathcal{H}) = \{\text{finite rank operators on } \mathcal{H}\}$

Indeed: Any element of $C(\mathcal{H})$ is of finite rank by definition. Reciprocally a rank-1 operator is of the form $A\xi = \langle \varphi, \xi\rangle \psi$ and we have the polarization identity ← notation: $A = |\psi\rangle\langle\varphi|$

$$\langle \varphi, \cdot\rangle \psi = \frac{1}{4}\sum_{k=0}^{3} i^k P_{\psi + i^k \varphi} \in C(\mathcal{H})$$

- Step 2. Let $M \subset \mathcal{H}$ be a finite dimensional subspace of $\mathcal{H}$. Then S maps $C(M)$ onto $C(M')$ where $M' \subset \mathcal{H}$ is so that $\dim(M') = \dim(M)$

Indeed: Let P_M be the orthogonal projector onto M. By $(*)$, SP_M is a projector of the same dimension as P_M. We set $M' = \mathrm{Ran}(SP_M)$ $\left(\begin{array}{l}\text{Note: } \mathrm{Ran}(P) \subset M \\ \Rightarrow \mathrm{Ran}(SP) \subset M'\end{array}\right)$

- Step 3. Let M, M' be as above with $\dim M = 2$. Then there is a (anti)linear isometry $U(M): M \to M'$

$$SP = UPU^*$$

By picking bases in M, M': $M \simeq M' \simeq \mathbb{C}^2$. Any 1-d projector π on $\mathbb{C}^2$ can be written as

(06)

$$\Pi = \frac{1}{2}(\mathbb{1} + \vec{\sigma}\cdot\vec{e}) \quad \text{for some } \vec{e} \in S^2$$

and where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

are the Pauli matrices.

Note: $\sigma_i \sigma_j = \delta_{ij}\mathbb{1} + i\,\varepsilon_{ijk}\sigma_k$

and so $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,\mathbb{1} + i\,\vec{\sigma}\cdot(\vec{a}\wedge\vec{b})$

In particular: + $\Pi^2 = \frac{1}{4}(\mathbb{1} + 2\vec{\sigma}\cdot\vec{e} + \vec{e}\cdot\vec{e}\,\mathbb{1}) = \Pi$ indeed

+ $\mathrm{Tr}(\Pi) = 1 \Rightarrow \Pi$ is one-dimensional.

We conclude that S induces a map on S^2. Moreover:
$e \mapsto e'$

$$\mathrm{Tr}(\Pi_1\Pi_2) = \frac{1}{2} + \frac{1}{2}\vec{e}_1\cdot\vec{e}_2$$

so that the equality $\mathrm{Tr}(\Pi_1\Pi_2) = \mathrm{Tr}((S\Pi_1)(S\Pi_2))$ implies

$$\vec{e}_1\cdot\vec{e}_2 = \vec{e}_1{}'\cdot\vec{e}_2{}'$$

$\Rightarrow$ $e \mapsto e'$ is a solid rotation R, $R(S)$ i.e. an element in $O(3)$

+ If $\det(R) = +1$ (i.e. $R \in SO(3)$), then

$$(R\vec{e})\cdot\vec{\sigma} = \vec{e}\cdot U(S)\vec{\sigma}U(S)$$

where $U(S) \in SU(2)$ (this will be done explicitely later)

in particular: $U(S)$ is unitary

+ If $\det(R) = -1$. Let R_0 be defined by

$$R_0\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}\alpha_1\\ -\alpha_2\\ \alpha_3\end{pmatrix}, \quad \det R_0 = -1$$

For any R: $\det(R) = -1$, the rotation $R' = RR_0$ belongs to $SO(3)$ and so it is unitarily implemented by $U(R')$.

It suffices to exhibit an antiunitary for R_0:

$$U\sigma_i U^* = \begin{cases} \sigma_i & (i=1,3) \\ -\sigma_2 \end{cases}$$

is given by the antiunitary $U\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$.

→ In all case $S\pi = U(S)\pi U(1)^*$ indeed. $(\diamond)$

- Step 4. General case. Let $\Pi \subset \mathcal{H}$ be s.t. $\dim \Pi = 2$, and let $\det(S\restriction_\Pi)$ denote the determinant just defined.

Claim: $\det(S\restriction_\Pi)$ is independent of Π.

Indeed: S being a symmetry: $\mathrm{Tr}((SP_1 - SP_2)^2) = \mathrm{Tr}((P_1-P_2)^2)$, namely: S is a continuous map on $P\mathcal{H}$, (w.r.t. $\|P-Q\|^2 = \mathrm{Tr}((P-Q)^2)$)

But then: If $\Pi(t)$ is a smooth family of 2-d subspaces of $\mathcal{H}$, we can choose a smooth family of bases so that $\det(S\restriction_{\Pi(t)})$ is continuous, hence constant, because $\mathcal{H}$ is connected.

→ Notation: $D(S) = \det(S\restriction_\Pi)$.

Now Construction of $U(S)$.

Pick $P \in P\mathcal{H}$ and a normalized $\psi = P\psi$.

Pick a normalized $\psi' = (SP)\psi'$

For any 2-dim Π s.t. $\psi \in \Pi$, let $U_\Pi(S)$ be the unitary of step 3. It satisfies

$$(SP)U_\Pi(S)\psi \overset{(\diamond)}{=} U_\Pi(S)(SP)\psi = U_\Pi(S)\psi$$

and we fix the phase of $U_\Pi(S)$ by imposing

$$U_\eta(S)\psi = \psi'$$

We now define $U(S)$ on all of $\mathcal{H}$ by setting, for any $\varphi \in \mathcal{H}$ that is linearly independent of ψ:

$$U(S)\varphi := U_\eta(S)\varphi \quad , \eta = \operatorname{span}\{\varphi, \psi\}.$$

It remains to check that $U(S)$ is an isometry.

$$S(|\psi\rangle\langle\varphi|) = U(S)|\psi\rangle\langle\varphi|U(S)^* = |\psi'\rangle\langle\varphi|U(S)^*$$

Hence:

$$\mathrm{Tr}\left((S|\psi\rangle\langle\varphi|)^*(S|\psi\rangle\langle\xi|)\right)$$
$$= \mathrm{Tr}\left(|U(S)\varphi\rangle\langle\psi'|\psi'\rangle\langle U(S)\xi|\right) = \langle U(S)\xi, U(S)\varphi\rangle$$

(we use: $\left(|\psi\rangle\langle\varphi|U(S)^*\right)\xi = \left(|\psi\rangle\langle\varphi|\right)U(S)^*\xi$
$$= \langle\varphi, U(S)^*\xi\rangle\psi = \langle U(S)\varphi, \xi\rangle\psi$$
$$= \left(|\psi\rangle\langle U(S)\varphi|\right)\xi$$

$$+ \quad \langle\mu, (|\psi\rangle\langle\varphi|)^*\nu\rangle = \langle(|\psi\rangle\langle\varphi|)\mu, \nu\rangle$$
$$= \overline{\langle\varphi,\mu\rangle}\langle\psi,\nu\rangle$$
$$\left.\langle\mu, |\varphi\rangle\langle\psi|\nu\rangle = \langle\mu,\varphi\rangle\langle\psi,\nu\rangle\right] =$$
so that $(|\psi\rangle\langle\varphi|)^* = |\varphi\rangle\langle\psi|$)

Hence:

$$\langle U(S)\xi, U(S)\varphi\rangle = \begin{cases} \mathrm{Tr}\left((|\psi\rangle\langle\varphi|)^*|\psi\rangle\langle\xi|\right) = \langle\xi,\varphi\rangle \\ \text{——— " ———} \quad = \overline{\langle\xi,\varphi\rangle} \end{cases}$$

by (1) on p. 64.

- Parity $U_P^\dagger x_i U_P = -x_i \; ; \; U_P^\dagger p_i U_P = -p_i$

$\Rightarrow U_P^\dagger [x_i, p_j] U_P = [x_i, p_j]$

and since the commutator $= i\delta_{ij}$, we conclude that U_P must be <u>unitary</u>

Time reversal $\quad U_T^\dagger x_i U_T = x_i \; ; \; U_T^\dagger p_i U_T = -p_i$

$\Rightarrow \quad U_T^\dagger [x_i, p_j] U_T = -[x_i, p_j]$

and so U_T must be antiunitary

In both cases: the square of the symmetry is identity

Hence $\quad U_T^2 = c_T \quad ; \quad U_P^2 = c_P$

with $|c_T| = |c_P| = 1$

+ unitary case: pick $\tilde{U}_P = \bar{c}_P^{1/2} U_P$ to have $\tilde{U}_P^2 = 1$

+ antiunitary case: doesn't work:

$\tilde{U}_T = \bar{c}_T^{1/2} U_T$ yields $\quad \tilde{U}_T^2 = \overline{c_T}^{1/2} c_T^{1/2} \cdot c_T$

However $\quad U^2 U = U U^2$

implies $\quad c_T = \bar{c}_T$ leaving just two possibilities.
$$\begin{cases} c_T = 1 \\ c_T = -1 \end{cases}$$

Example. On $L^2(\mathbb{R})$, $\quad (U_P \psi)(x) = \psi(-x)$
$(U_T \psi)(x) = \overline{\psi(x)}$

with $c_T = +1$

Later: with spin j: $\quad c_T = (-1)^{2j}$

- Symmetry of the dynamics:

$$U e^{-itH} = \begin{cases} e^{-itH} U & \text{(unitary)} \\ e^{+itH} U & \text{(anti-unitary)} \end{cases}$$

by taking $\left.\frac{d}{dt}\right|_{t=0}$: $\quad \left.\begin{matrix} -iUH \\ iUH \end{matrix}\right\} = \begin{cases} -iHU \\ +iHU \end{cases}$

$\Rightarrow \quad [U, H] = 0 \quad$ (symmetry)

- Continuous symmetries: Let G be a compact connected Lie group.

Def: Projective representation of G in $\mathcal{H}$ is a symmetry $S_g : P\mathcal{H} \to P\mathcal{H} \quad \forall g \in G$

s.t. $S_g \circ S_h = S_{gh}$

and $g \mapsto S_g(P)$ is continuous

Let U_g be the corresponding (anti)unitary. Then

$$U_g U_h P U_h^\dagger U_g^\dagger \overset{!}{=} U_{gh} P U_{gh}^\dagger \quad \forall \text{ projectors } P \text{ in } \mathcal{H}$$

which implies

(i) Since any $g \in G$ is $g = h^2$ (by exponential map)
U_g must be unitary, not antiunit.

(ii) $U_g U_h = \omega(g,h) U_{gh}$ also called a projective representation.
with $\omega(g,h) \in U(1)$

(iii) By considering associativity with g, h, l
$\omega(g,h)\,\omega(gh,l) = \omega(g,hl)\,\omega(h,l)$

Note: U_g is determined up to a phase. Redefining
$U_g \to U'_g = \mu_g U_g$ $(\mu_g \in U(1))$ yields.

$\omega'(g,h) = U'_g U'_h U'^{-1}_{gh} = U_g U_h U^{-1}_{gh}\,\mu_g \mu_h \mu_{gh}^{-1}$
$= \omega(g,h)\,\mu_g \mu_h \mu_{gh}^{-1}$

and U'_g is a "usual" representation when
$\omega'(g,h) = 1$.

- From an abstract point of view: (Abelian group)

Def: n-cochain $f: G^n \to U(1)$
Form a group $C^n(G, U(1))$
+ fg: pointwise multiplication
+ e: $(g_1, \dots, g_n) \mapsto 1$
+ $f^{-1}(g_1, \dots, g_n) = f(g_1, \dots, g_n)^{-1}$

Now: Coboundary operator:
$\delta^n: C^n(G, U(1)) \to C^{n+1}(G, U(1))$
$(\delta^n f)(g_1, \dots, g_{n+1}) = f(g_2, \dots, g_{n+1}) \cdot \prod_{i=1}^{n} f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})^{(-1)^i} \cdot (f(g_1, \dots, g_n))^{(-1)^{n+1}}$

and we let $B^n(G, U(1)) = \mathrm{Im}(\delta^{n-1})$ "coboundary"
$Z^n(G, U(1)) = \mathrm{Ker}(\delta^n)$ "cocycle"

Fact: $\delta^{n+1} \circ \delta^n f = e$
Hence $B^n(G, U(1)) \subset Z^n(G, U(1))$
(in fact a subgroup)

Def: n-cohomology group: $H^n(G, U(1)) = Z^n(G, U(1)) / B_n(G, U(1))$

Case $n=2$

* $f \in Z^2(G, U(1))$ if $\delta f = 1$ i.e.
  $f(h,\ell)\, f(gh,\ell)^{-1}\, f(g,h\ell)\, f(g,h)^{-1} = 1$
  ⤳ ω arising from a projective representation is a cocycle
* $f \in B^2(G, U(1))$ if $\exists\, \lambda$ s.t.
  $f(g,h) = \lambda(g)\, \lambda(g,h)^{-1}\, \lambda(h)$
* Two cocycles f, f' are cohomologous if (they differ by a coboundary, namely)
  $f'(g,h) = f(g,h)\, \lambda(g)\, \lambda(h)\, \lambda(g,h)^{-1}$

⤳ projective representations differing by a redefinition of the phase of U_g have "cohomologous" cocycles.

U_g is equivalent to a representation iff $[\omega] = [1]$

⤳ $H^2(G, U(1))$ classifies projective representations.

Fact for later use:

$$H^2(SU(2), U(1)) = \{1\}.$$

* We now consider spatial rotations, $G = SO(3)$

We start with a unitary representation $U(R)$, $R \in SO(3)$

Ex: on $L^2(\mathbb{R}^3)$:

$$(U_0(R)\psi)(x) = \psi(R^{-1}x)$$

$SO(3)$ is a smooth manifold. Elements of its tangent space at $\mathbb{1}$ are given by

$$\Omega = \frac{d}{dt} R(t)\Big|_{t=0}$$

"infinitesimal rotations"

smooth curve in $SO(3)$, with $R(0) = \mathbb{1}$.

Product in the tangent space:

$$[\Omega_1, \Omega_2] = \frac{d}{dt} R_1(t)\, \Omega_2\, R_1(t)^{-1}\Big|_{t=0}$$

$$= \frac{d}{dt} R_1(t) \frac{d}{ds} R_2(s)\Big|_{s=0} R_1(t)^{-1}\Big|_{t=0}$$

$\leadsto$ The tangent space is the Lie algebra $so(3)$

Characterization:

$\mathcal{D}\, R(t)^T R(t) = \mathbb{1} \Leftrightarrow \Omega^T + \Omega = 0$

(for $\Leftarrow$): $R(t) := e^{\Omega t}$.

$so(3) = \{$antisymmetric (real) matrices$\}$. We write:

$$\Omega(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

namely $\Omega(\omega)\vec{x} = \vec{\omega} \wedge \vec{x}$

Check: $[\Omega(\vec{\omega}_1), \Omega(\omega_2)] = \Omega(\omega_1 \wedge \omega_2)$

and so $[\Omega_1, \Omega_2] = \Omega_3$

where $\Omega_i = \Omega(e_i)$.

Now: if $U(R)$ is an unitary rep of $SO(3)$, then $U(\Omega) = \frac{d}{dt} U(R(t))|_{t=0}$ defines a representation of $so(3)$

(clear if $\dim \mathcal{H} < \infty$; otherwise $U(\Omega)$ is in general unbounded, defined as in Stone's theorem)

Check homomorphism:

If $U(R)^+ U(R) = \mathbb{1}$, then $U(\Omega)^+ + U(\Omega) = 0$

$\hookrightarrow$ Define: Angular momentum operator

$$M(\vec{\omega}) = i U(\Omega(\vec{\omega}))$$

+ $M(\vec{\omega})$ is self-adjoint

+ $M(\vec{\omega}) = \sum_i M_i \omega_i$ where $\underline{[M_1, M_2] = i M_3}$

Lie algebra $so(3)$

We now classify the irreducible representations of $so(3)$

$\{0\}$ and $\mathcal{H}$ are only inv. subspaces

<u>Theorem</u>: The irreducible unitary representations of $so(3)$ are given by D_j : $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
where
$\dim D_j = 2j+1$
and
$$\sum_{i=1}^{3} U_j(\Pi_i)^2 = j(j+1)\mathbb{1}$$

<u>Note</u>: As above $U(\Pi_i)$ are actually self-adjoint matrices (despite the standard name "unitary")

<u>Examples</u>: + $j=0$: trivial : $\Pi_i = 0$ (drop U_j!)
+ Fundamental representation on $\mathcal{H} = \mathbb{C}^3$
$U(R) = R$; $U(\Omega) = \Omega$
+ Adjoint representation on $\mathcal{H} = so(3)$:
$U(R)\Omega = R\Omega R^{-1}$
$U(\Omega_1)\Omega_2 = [\Omega_1, \Omega_2]$
+ Exercise: spherical harmonics.

Proof. Let $\Pi_\pm = \Pi_1 \pm i\Pi_2$. Then

$$[\Pi_3, \Pi_\pm] = \pm \Pi_\pm \quad ; \quad [\Pi_+, \Pi_-] = 2\Pi_3$$

equivalently $\Pi_3 \Pi_\pm = \Pi_\pm (\Pi_3 \pm \mathbb{1})$

Note. $\Pi^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2$ commutes with everything
$[\Pi^2, \Pi_i] = 0$
$\Pi_\pm^* = \Pi_\mp$

Let λ be an eigenvalue of Π^2. The corresponding eigenspace is invariant under the action of the representation, it is not $\{0\}$, hence it must be all of $\mathcal{H}$, namely
$\Pi^2 = \lambda \mathbb{1}$

Let ψ be $\Pi_3 \psi = m\psi$, $m \in \mathbb{R}$.
If $\Pi_\pm \psi \neq 0$, then $m \pm 1$ is also an eigenvalue:
$$\Pi_3 \Pi_\pm \psi = \Pi_\pm (\Pi_3 \pm \mathbb{1})\psi = (m \pm 1)\Pi_\pm \psi$$
and ψ is an eigenvector of $\Pi_\mp \Pi_\pm$:
$$\Pi_\mp \Pi_\pm \psi = (\Pi^2 - \Pi_3(\Pi_3 \pm \mathbb{1}))\psi$$
$$= (\lambda - m(m \pm 1))\psi \qquad (*)$$
The recursively constructed eigenvectors span an invariant subspace of $\mathcal{H}$ hence $\mathcal{H}$ itself by irreducibility.

Now
$$\langle \psi, \Pi_\mp \Pi_\pm \psi \rangle = \|\Pi_\pm \psi\|^2 \geq 0$$
$$\parallel$$
$$\lambda - m(m \pm 1)$$

with m increasing, this eventually becomes negative. Hence there can be only a finite number of values of m

$\Rightarrow \dim \mathcal{H} < \infty$

and there must be j, Ψ_j s.t. $\|\Psi_j\| = 1$

$\Pi_3 \Psi_j = j \Psi_j$ ← for now, $j \in \mathbb{R}$.

$\Pi_+ \Psi_j = 0$

and from $(*)$: $0 = (\lambda - j(j+1))\Psi$ i.e.

$\lambda = j(j+1)$

Now: for $m \leq j$, we let, recursively

$\Pi_- \Psi_m =: c_m \Psi_{m-1}$ $\quad m = j, j-1, \dots$

as long as $\Pi_- \Psi_m \neq 0$

and recall $c_m \Pi_3 \Psi_{m-1} = \Pi_3 \Pi_- \Psi_m$

$= (m-1)\Pi_- \Psi_m$

$= c_m (m-1) \Psi_{m-1}$

i.e. $\Pi_3 \Psi_{m-1} = (m-1)\Psi_{m-1}$

The recursion stops whenever $\Pi_- \Psi_{m_0} = 0$, namely

$0 = \langle \Psi_{m_0}, \Pi_+ \Pi_- \Psi_{m_0} \rangle = j(j+1) - m_0(m_0 - 1)$

$= (j + m_0)(j + 1 - m_0)$

i.e. $m_0 = -j$ $\quad (m_0 = j+1 > 0$ is excluded$)$

$\Rightarrow$ Since $j, j-1, \dots, -(j-1), -j$ are integer spaced,

$j - (-j) = 2j \in \mathbb{N}$, hence $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

For j given, the vectors

$\{\Psi^{(j)}_m, \ m = -j, \dots, j\}$

form an ON basis of $\mathcal{D}_j$ for which

$$\begin{cases} \Pi^2 \Psi^{(j)}_m = j(j+1)\Psi^{(j)}_m \\ \Pi_3 \Psi^{(j)}_m = m \Psi^{(j)}_m \\ \Pi_\pm \Psi^{(j)}_m = \left(j(j+1) - m(m \pm 1)\right)^{1/2} \Psi^{(j)}_{m \pm 1} \end{cases} \qquad \square$$

"raising" and "lowering" operators

- We have obtained the irreps of $so(3)$. If the rep. arises from one of $SO(3)$, then

$U(R(\vec{e}_3, \varphi)) = \exp(-i \Pi_3 \varphi)$

and $R(\vec{e}_3, 2\pi) = id$ implies

$\exp(-2\pi i m)\Psi^{(j)}_m = \Psi^{(j)}_m$, namely $m \in \mathbb{Z}$

In turn, this means $j \in \mathbb{Z}$
This also means that, if j is a half-odd integer, then only 4π-rotations will be trivial, while $\exp(-2\pi i T_3)\Psi^{(j)}_m = -\Psi^{(j)}_m$!

Claim The latter irreps arise from unitary representations of $SU(2)$ (the "quantum mechanical" rotation group)

$SU(2)$: complex 2×2 matrices s.t.
$V^\dagger V = VV^\dagger = \mathbb{1}$ and $\det V = 1$
tangent space $\Theta = \frac{d}{dt} V(t)|_{t=0}$ s.t.
$\Theta^\dagger + \Theta = 0$, $\mathrm{Tr}(\Theta) = 0$ (because $\det V = \exp(\mathrm{Tr}(\log V))$)
Lie algebra $su(2)$ with product $[\Theta_1, \Theta_2] = \Theta_1\Theta_2 - \Theta_2\Theta_1$
General $\Theta = \Theta(\vec{v}) = -\frac{i}{2}\begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}$
which can be written with Pauli matrices: $\Theta = -\frac{i}{2}\vec{\sigma}\cdot\vec{v}$
$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(correspondingly: $\Theta_1, \Theta_2, \Theta_3$ is a basis for $su(2)$ as a 3-dim real vector space)
Again: $[\Theta(\vec{v}_1), \Theta(\vec{v}_2)] = \Theta(\vec{v}_1 \wedge \vec{v}_2)$
or
$[\Theta_1, \Theta_2] = \Theta_3$
so we have:
Lemma: $su(2)$ and $so(3)$ are isomorphic
Proof: The isomorphism is given by $\Theta(\vec{\omega}) \mapsto \Omega(\vec{\omega})$ □
Hence: The irreducible representations of $su(2)$ are those of $so(3)$.
While again: every representation of $SU(2)$ yields one of $su(2)$, the reciprocal holds too (unlike $SO(3)$!)

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Theorem Every representation $\mathcal{D}_j$, $j = 0, \frac{1}{2}, 1, \dots$ of $su(2)$ corresponds to a representation of $SU(2)$, for which

$$U_j(-V) = (-1)^{2j} U_j(V)$$

(Proof) * Obvious for the trivial rep $j=0$.
* For $j = \frac{1}{2}$ we set for $V = e^{At}$

$$U_{1/2}(V) = e^{At}$$

which is the fundamental rep of $SU(2)$ and for which $U_{1/2}(-V) = -e^{At} = -U_{1/2}(V)$
* Then if the claim holds for j, then
$U(V) := U_j(V) \otimes V$
is a representation on $\mathcal{D}_j \otimes \mathcal{D}_{1/2}$
s.t. $U(-V) = (-1)^{2j+1} U(V)$
Since $\mathcal{D}_j \otimes \mathcal{D}_{1/2} = \mathcal{D}_{j-\frac{1}{2}} \oplus \mathcal{D}_{j+1/2}$, the second block yields $U_{j+1/2}(V)$ $\square$

- Every rep of $su(2)$ yields a rep of $SU(2)$. In particular $SO(3)$ is a rep of $SU(2)$
$R: SU(2) \to SO(3)$
$V \mapsto R = R(V)$
where $V = e^{\Theta(\bar{\omega})t} \mapsto R = e^{\Omega(\bar{\omega})t}$
this is surjective
but not injective. In fact $R(V) = R(-V)$
Proof Check $[\Theta(\bar{\omega}), \Theta(\bar{a})] = \Theta(\bar{\omega} \wedge \bar{a})$
calculation $= \Theta(\Omega(\bar{\omega})\bar{a})$
Exponentiating this:
$$V \Theta(\bar{a}) V^* = \Theta(R(V)\bar{a})$$
Hence $V \Theta V^* = \Theta \quad \forall \Theta \in su(2)$
$\Leftrightarrow V = \pm \mathbb{1}$ (namely $R(V) = id$) $\square$

Conclusion: $SO(3) = SU(2)/\{\pm 1\}$

In fact, the irreps with j half-odd integer are *projective* representations of $SO(3)$

- The spin of the electron

N electrons in a constant B-field $\vec{B} = B\vec{e}_3$

$$H = \sum \frac{1}{2m}\left(\vec{p}_i - \frac{e}{c}\vec{A}(\vec{x}_i)\right)^2 + V(\vec{x}_1, \dots, \vec{x}_N) \quad \text{on } L^2(\mathbb{R}^{3N})$$

$$\vec{A} = \frac{1}{2}(\vec{B} \wedge \vec{x}) \quad \text{i.e.} \quad \vec{B} = \vec{\nabla} \wedge \vec{A}$$

Rotation-inv. potential $V(\vec{x}_1, \dots, \vec{x}_N) = V(R\vec{x}_1, \dots, R\vec{x}_N)$

Weak field: keep only terms linear in B:

$$\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x})\right)^2 = p^2 - \frac{e}{c}(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + O(B^2)$$
$$= p^2 - \frac{e}{c}\vec{B}\cdot(\vec{x} \wedge \vec{p}) + O(B^2)$$

$$\rightarrow \quad H = H_0 + \mu_B B M_3 \qquad \left(\mu_B = \frac{e}{2mc}\right)$$

unperturbed atom $\qquad$ here $\vec{M} = \sum_{i=1}^{N} \vec{x}_i \wedge \vec{p}_i$

H_0 is invariant under rotations represented by
$$(U_0(R)\psi)(\vec{x}_1, \dots, \vec{x}_N) = \psi(R^{-1}\vec{x}_1, \dots, R^{-1}\vec{x}_N)$$
on $L^2(\mathbb{R}^{3N})$, $R \in SO(3)$

Now: If E_0 is a degenerate eigenvalue of H_0, the corresponding eigenspace carries a rep of $SO(3)$ — in general irreducible. As a subrep of U_0, it would have integer j, with degeneracy $2j+1$ $(j = 0, 1, 2, \dots)$

For H this implies:

$$H\psi^{(j)}_m = (E_0 + \mu_B B m)\psi^{(j)}_m \qquad (m = -j, \dots, j)$$

namely the spectrum splits:

$$E_0 \;\text{———}\; \rightsquigarrow \; \equiv \Big\} \; 2j+1$$

with gaps proportional to B, independent of N and E_0.

Experimental fact (Zeeman effect)

* $2j+1$ is even whenever N is odd
* the splitting not indep of N, E_0

Reason the electron carries Spin, namely the Hilbert space for one electron is

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \simeq L^2(\mathbb{R}^3; \mathbb{C}^2)$$

which carries a representation of $SU(2)$

$$U(V) = U_0(R(V)) \otimes V$$

in $D_{1/2}$

The spin matrices are given by $M_j = \frac{1}{2}\sigma_j$,

with $M_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

and $\psi^{(1/2)}_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |\uparrow\rangle$, $\psi^{(1/2)}_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |\downarrow\rangle$

"Spin up" "Spin down"

↳ total angular momentum:

$$\vec{J} = (\vec{x} \wedge \vec{p}) \otimes \mathbb{1} + \mathbb{1} \otimes \vec{M}$$

'orbital' angular mom. spin

• Note: This is how migratory birds navigate!

- Back to the stability of matter

Magnetic kinetic energy with magnetic field and spin: (Pauli) on $L^2(\mathbb{R}^3, \mathbb{C}^2)$

$$T_A(\vec{p}) = \frac{1}{2}\left|\vec{\sigma}\cdot\left(\vec{p} + \sqrt{\alpha}\,\vec{A}(\vec{x})\right)\right|^2 \geq 0$$

We expand, recalling that $\sigma_i\sigma_j = \sum_k i\varepsilon_{ijk}\sigma_k$ $(i \neq j)$, $\sigma_i^2 = \mathbb{1}$

$$T_A(\vec{p}) = \frac{1}{2}\left(\vec{p} + \sqrt{\alpha}\vec{A}(\vec{x})\right)^2 + \frac{1}{2}\sum_{i,j,k}\sqrt{\alpha}\, i\varepsilon_{ijk}\sigma_k\left(p_iA_j + A_ip_j\right)$$

symmetrical in $i \leftrightarrow j$ so vanishes

$$= -i(\partial_i A_j) + (A_jp_i + A_ip_j)$$

$$= \frac{1}{2}\left(\vec{p} + \sqrt{\alpha}\vec{A}(\vec{x})\right)^2 + \frac{\sqrt{\alpha}}{2}\vec{\sigma}\cdot\vec{B} \qquad (\vec{B} = \nabla\wedge\vec{A})$$

oblivious to spin, positive — dependent on spin, no definite sign

We shall include the field energy, given by

$$E(\vec{B}) = \frac{1}{8\pi}\int_{\mathbb{R}^3}|B(\vec{x})|^2\,dx$$

A state $\psi \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ is called a zero-mode if $\langle\psi, T_A(\vec{p})\psi\rangle = 0$, or equivalently

$$\sigma\cdot\left(\vec{p} + \sqrt{\alpha}\vec{A}(\vec{x})\right)\psi(\vec{x}) = 0$$

Note: If $A = 0$, then a zero mode is a harmonic function, and so there are no zero modes.

Fact: zero modes exist: there are explicit examples (see Loss–Yau)

With this in mind, we consider the one-particle problem, namely

$$H_A := T_A(\vec{p}) - \frac{Z\alpha}{|x|} + E(\vec{B})$$

If ψ is a zero mode for A, so is ψ_λ for A_λ, $\lambda > 0$ with

$$\psi_\lambda(\vec{x}) = \lambda^{3/2}\psi(\lambda\vec{x}) \quad ; \quad \vec{A}_\lambda(\vec{x}) = \lambda\vec{A}(\lambda\vec{x})$$

chosen so that (check λ_0 calculation)

$$\|\psi_\lambda\|_2 = \|\psi\|$$

But then $\vec{B}_\lambda(\vec{x}) = \lambda^2 \vec{B}(\lambda x)$ and so
$E(B_\lambda) = \lambda E(B)$

Hence

$$\langle \psi_\lambda, H_A \psi_\lambda \rangle = -Z\alpha \langle \psi_\lambda, |x|^{-1}\psi_\lambda \rangle + \lambda E(B)$$

$$= \lambda^3 \int \frac{|\psi(\lambda x)|^2}{|x|}dx = \lambda^3 \cdot \lambda \cdot \lambda^{-3} \int \frac{|\psi(x)|^2}{|x|}dx$$

$$= \lambda\left(-Z\alpha \langle \psi, |x|^{-1}\psi \rangle + E(B)\right)$$

Conclusions: In the presence of a zero mode:

(i) If $E(B)$ is not taken into account, the atom is unstable

(ii) Even with $E(B)$ stability can only hold for α small enough (as compared to Z)

Theorem If $Z\alpha^2 \leq \frac{\pi}{2}\left(\frac{3}{4}\right)^{3/2}$,

then $\langle \psi, H_A \psi \rangle \geq -(Z\alpha)^2 \|\psi\|^2$

For any $\psi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ and any $\vec{A}(\vec{x})$ such that $\vec{B} \in L^2(\mathbb{R}^3)$

Note: For the physical $\alpha = 1/137$, we have the bound $Z \leq 19\,160$