A)

Foundational questions

- QM provides, in general, only probabilities for an event to happen, such as that the result of a measurement of the 3rd component of the spin of an electron is "up".

→ Question: (Einstein-Podolski-Rosen, 1935) Can Quantum mechanical description of physical reality be considered complete?

+ Element of physical reality: "if we can predict with certainty the value of the corresponding physical quantity."
+ Theory is complete if every element of physical reality has a counterpart in physical theory.

⇒ Q.M is not complete. Indeed:

Two-spin-$\frac{1}{2}$ particles: $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$

State $\Psi = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$ "entangled"

where $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

in the basis where $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(example: state of two electrons after decay of a pion)

Make a measurement of both spins. The two possible results are:

B]

$(+\frac{1}{2}, -\frac{1}{2})$ or $(-\frac{1}{2}, +\frac{1}{2})$

In particular: the measurement of S_1^3 can be predicted with certainty by reading off the measurement of S_2^3

namely: S_1^3 is an element of physical reality, but it is unknown before measuring S_2^3.

In fact: the same holds for (S_1^j, S_2^j) $(j=1,2)$

According to QM: An observable has a deterministic value in a state ψ only if ψ is an eigenvector.

But: Since $[S_1^1, S_1^2] = i S_1^3$, there is no joint eigenvector

$\Rightarrow$ QM is incomplete.

- Question: Can QM be "completed" such that the new theory
  - (i) Reproduces the prediction of QM or
  - (ii) Does not, in situations which could be tested by experiments?

- Hidden variable theories.

Theory where all values of observables are determined, although the state may not be known, namely it may only give a probability of these values c)

QM is not of that type since ψ gives only the probability distribution of the values of a given observable being measured.

Two possibilities: $(\dim \mathcal{H} < \infty)$

* HV1) $\exists$ probability space Ω ($\omega \in \Omega$ are the "hidden variables") and two maps:

(i) $\rho : \mathcal{H} \longrightarrow \mathcal{P}(\Omega)$ (prob. measures)

$\qquad \psi \longmapsto \rho_\psi$

(ii) $F : \underset{\text{self-adjoint}}{\mathcal{B}(\mathcal{H})} \longrightarrow \mathcal{M}(\Omega)$ (real-valued measurable functions)

s.t.

If $A = \sum_j a_j P_j$ is the spectral decomp. of $A = A^\dagger$,

then $\quad \langle \psi, P_j \psi \rangle = \rho_\psi\left((F(A))^{-1}(a_i) \right)$

(i.e. $F(A)(\omega)$ is the value of A in $\omega \in \Omega$).

In particular: the expectation value of A in ψ is given by

$$\langle \psi, A\psi \rangle = \sum_j a_j \, \rho_\psi\left((F(A))^{-1}(a_i) \right)$$

$$= \int_\Omega F(A)(\omega) \, d\rho_\psi(\omega) \quad \text{(Lebesgue integral)}$$

D]

* HV2) As HV1), with (ii) replaced by a map $\chi : (\text{projectors } P = P^\dagger = P^2 \text{ in } \mathcal{H})$
$$\longrightarrow \chi(P) \subset \Omega \quad \text{measurable set in } \Omega$$
s.t. $\langle \psi, P\psi \rangle = \rho_\psi(\chi(P))$.
and for any $\{P_j\} : \sum_j P_j = \mathbb{1}$ s.t. $P_i P_j = \delta_{ij} P_j$, $\{\chi(P_i)\}$ is a partition of Ω.

Note: HV2) implies HV1): For $A = \sum_j a_j P_j$

Define $F(A)(\omega) = \sum_j a_j \underbrace{\chi(P_j)(\omega)}$ understood as the characteristic function of $\chi(P_j)$

indeed $(F(A))^{-1}(a_j) = \chi(P_j)$ and so
$$\langle \psi, P_j \psi \rangle = \rho_\psi(\chi(P_j))$$
$$= \rho_\psi\left((F(A))^{-1}(a_j)\right)$$

⇝ Difference: A projector P $(\neq 0, \mathbb{1})$ is in general part of different decompositions of $\mathbb{1}$ (except if $\dim \mathcal{H} = 2$!) If two observable $A, \tilde{A}$ have the same eigenvecter for eigenvalues $a, \tilde{a}$, the sets
$$(F(A))^{-1}(a) \quad , \quad (F(\tilde{A}))^{-1}(\tilde{a})$$
may be different in HV1, but the set

E) $X(P)$ associated with P is fixed.

HV1 is called <u>contextual</u>

HV2 is not.

(*)

- <u>Example</u> : Spin $\frac{1}{2}$: Both HV1, HV2 are possible.

$$\Omega = \{\omega = (\psi, \lambda) \cdot \psi \in \mathbb{C}^2, \|\psi\| = 1, \lambda \in [-1, 1]\}$$

$$d\rho_\psi(\omega) = \frac{d\lambda}{2}$$

- and for observables $\vec{\sigma} \cdot \vec{e}$ $(|\vec{e}| = 1)$

$$(\vec{\sigma} \cdot \vec{e})(\psi, \lambda) = \begin{cases} 1 & \text{if } \lambda \in [-\langle \psi, (\vec{\sigma} \cdot \vec{e})\psi \rangle, 1] \\ -1 & \text{otherwise} \end{cases}$$

Hence: $\rho_\psi\left((\vec{\sigma} \cdot \hat{e})^{-1}(\{\pm 1\})\right) = \frac{1}{2}\left(1 \pm \langle \psi, (\vec{\sigma} \cdot \vec{e})\psi \rangle\right)$

reproduces the probabilities of $P_\pm$ associated with the two eigenvalues ± 1 in QM. ✓

- and for projectors $P(\vec{e}) = \frac{1}{2}(1 + \vec{\sigma} \cdot \hat{e})$

$$X(P) = \begin{cases} [-\langle \psi, (\vec{\sigma} \cdot \vec{e})\psi \rangle, 1] & \text{if } \vec{e} \text{ is in upper half sphere} \\ [-1, \langle \psi, (\vec{\sigma} \cdot \vec{e})\psi \rangle] & \text{otherwise} \end{cases}$$

- <u>Theorem</u> (Kochen-Specker, 67) If $\dim \mathcal{H} \geq 3$, then HV2 is impossible (i.e. incompatible with QM)

<u>Proof</u> in case $\dim \mathcal{H} = 8$, hence ≥ 8.

Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

and let $\sigma^j_1 = \sigma^j \otimes \mathbb{1} \otimes \mathbb{1}$

$\sigma^j_2 = \mathbb{1} \otimes \sigma^j \otimes \mathbb{1}$ $\quad \sigma^j_3 = \mathbb{1} \otimes \mathbb{1} \otimes \sigma^j$

E']

⊛ In HV2:

(i) Since $f(A) = \sum_j f(a_j) P_j$, we must have

$$F(f(A))(\omega) = \sum_j f(a_j)\, \chi(P_j)(\omega) = f(F(A)(\omega))$$

(ii) If $[A_1, A_2] = 0$, then $A_1 A_2$ can be diagonalized in the same basis and so $A_j = f(A)$ where A is another self-adjoint operator that is diagonal in the same basis. Hence

$$(A_1 A_2)(\omega) = F(A_1)(\omega) F(A_2)(\omega) \qquad (\diamond)$$

f)

Consider:

$$O_1 = \sigma^1 \otimes \sigma^2 \otimes \sigma^2$$
$$O_2 = \sigma^2 \otimes \sigma^1 \otimes \sigma^2$$
$$O_3 = \sigma^2 \otimes \sigma^2 \otimes \sigma^1$$

Note: $O_1 O_2 O_3 = -\sigma^1 \otimes \sigma^1 \otimes \sigma^1$ (since $(\sigma^2)^2 = \mathbb{1}$, $\sigma^1 \sigma^2 = -\sigma^2 \sigma^1$)

and $[O_i, O_j] = 0$

Since the factors within each tensor factor commute:

by $(\diamond)$: $f(O_1)(\omega) = f(\sigma_1^1)(\omega) f(\sigma_2^2)(\omega) f(\sigma_3^2)(\omega)$

and

$$(O_1 O_2 O_3)(\omega) = f(O_1)(\omega) f(O_2)(\omega) f(O_3)(\omega)$$
$$= f(\sigma_1^1)(\omega) f(\sigma_2^1)(\omega) f(\sigma_3^1)(\omega)$$
$$= f(\sigma^1 \otimes \sigma^1 \otimes \sigma^1)(\omega) \quad (*)$$

since $(f(\sigma^2)(\omega))^2 = f(\sigma^2)^2(\omega) = \mathbb{1}(\omega) = 1$

Now: $\langle \psi, O_1 O_2 O_3 \psi \rangle = -\langle \psi, \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \psi \rangle$

but $(*)$ and the rule of HVI:

$$\langle \psi, O_1 O_2 O_3 \psi \rangle = + \langle \psi, \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \psi \rangle$$

contradiction. $\blacksquare$

- To exclude HVI, we must assume additionally:

(L) (for <u>local</u>): If A, B correspond to space like (i.e. causally disconnected) measurements, then $f(AB)(\omega) = f(A)(\omega) f(B)(\omega)$

see $(\diamond)$

G)

- **Theorem** (Bell, 64): (HVI, L) is impossible
  Similar proof. insert $\boxed{J}$

- The 2022 Nobel Prize in Physics:
  "For experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"
  (Aspect, Clauser, Zeilinger)

- The GHZ game:
  + Three player, one referee
  + The referee gives each player a letter X or Y, in either combination XXX, XYY, YXY, YYX
  + The players can conspire before they recive their letter but cannot communicate afterwords.
  + Each shouts +1 or -1,
    They win if the product of what they shout is
    $$\begin{cases} +1 & \text{if they were given XXX} \\ -1 & \text{otherwise} \end{cases}$$

Classically: Let A_X, A_Y be what player A shouts if given X, Y. If there is a winning strategy:

$$A_X B_X C_X = 1$$
$$A_X B_Y C_Y = -1$$
$$A_Y B_X C_X = -1$$
$$A_Y B_Y C_X = -1$$

Contradiction: multiplying all equations yields
$+1 = -1$ ↯

Note: Even probabilistic and conspiratory strategies don't provide a way to always win the game.

In QM: Consider the state $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)$

(GHZ state, entangled)

on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

player A B C

Each player measures σ^1 if given X
σ^2 if given Y

and shouts ± 1 if the outcome is ± 1.

A trivial (but a little tedious calculation) yields that they always win!

Why! Because the result of the measurements, while each random are correlated
(e.g. if A measures z, A obtains ± 1 each with prob. $\frac{1}{2}$, but if it obtains $+1$ then A knows that the others also obtain $+1$, corresponding to $\uparrow$)

I)

Check: Eigenstates of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, namely $\frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle) = |\pm\rangle$.

Probability to obtain $\pm, \pm, \pm$:

$$\frac{1}{2^{3/2}}\Big(\underbrace{\langle \pm| \otimes \langle \pm | \otimes \langle \pm|}\Big)\frac{1}{\sqrt{2}}\big(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\big)$$

8 terms, only non-vanishing are $+\langle\uparrow\uparrow\uparrow|$ and $(-1)^{\#|-\rangle}\langle\downarrow\downarrow\downarrow|$

$\rightsquigarrow$ cancellation if the number of $|-\rangle$ is odd

$\rightarrow$ the three players will only measure combinations with an <u>even</u> number of "$-$"

$\Rightarrow$ they will shout a joint $+1$ with certainty if given XXX.

Eigenstates of $\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$: $\frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\downarrow\rangle)$

If two players receive a Y, they obtain $+\langle\uparrow\uparrow\uparrow|$ and $i^2(-1)^{\#|-\rangle}\langle\downarrow\downarrow\downarrow|$

$\rightsquigarrow$ cancellation if the number of $|-\rangle$ is even! the three player will only measure combinations with an odd number of "$-$" $\rightsquigarrow$ they will always shout a joint -1 with certainty if given XYY, YXY, YYX.

$\rightarrow$ Entanglement saves them!

11

- The Bell inequality applies to any local theory of hidden variables, with expectation value given by

$$\langle A \rangle_{\rho} = \int_{\Omega} A(\omega)\, d\rho(\omega) \qquad \text{(we write } A(\omega) \text{ for } (f(A))(\omega))$$

equivalence with QM not imposed.

Locality: A, B space-like separated $\Rightarrow$ $(AB)(\omega) = A(\omega)B(\omega)$

<u>Theorem</u> (Bell 64, Clauser 69). Let A, A' be space-like separated from B, B', taking values $\{\pm 1\}$. Then

$$\left| \langle AB \rangle + \langle A'B \rangle + \langle AB' \rangle - \langle A'B' \rangle \right| \leq 2. \qquad (*)$$

Proof (pure probability, and trivial)

We note that: if $A(\omega) = A'(\omega) \Rightarrow$ $A(\omega) + A'(\omega) = \pm 2$
$A(\omega) - A'(\omega) = 0$

if $A(\omega) = -A'(\omega) \Rightarrow$ $A(\omega) + A'(\omega) = 0$
$A(\omega) - A'(\omega) = \pm 2$

Hence, the expression on the l.h.s. of $(*)$ is

$$\big(A(\omega) + A'(\omega)\big)B(\omega) + \big(A(\omega) - A'(\omega)\big)B'(\omega)$$

equal to ± 2. $\boxtimes$

- Claim: QM is incompatible with the theorem: there are observables and a state ψ for which the l.h.s. of $(*)$ is equal to $2\sqrt{2} > 2$.

Indeed $A = (\vec{\sigma} \cdot \vec{e}_1)_{(1)}$; $A' = (\vec{\sigma} \cdot \vec{e}_1{}')_{(1)}$ on $\mathcal{H}_{(1)}$

k)

$B = (\vec\sigma\cdot\vec e_2)_{(1)} \qquad B' = (\vec\sigma\cdot\vec e_2{}')_{(2)}$ on $\mathcal{H}_{(2)}$

State: EPR pair $\Psi = \frac{1}{\sqrt2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

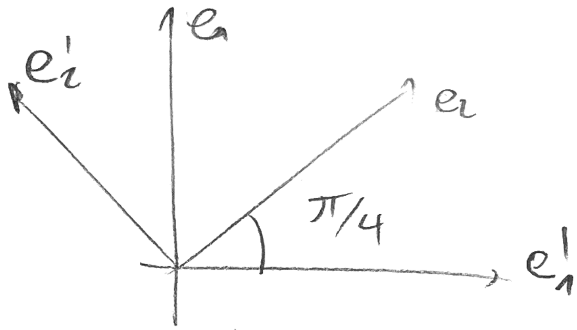
Check: $(\vec\sigma_{(1)} + \vec\sigma_{(2)})\cdot\vec e\,|\Psi\rangle = 0 \quad \forall \vec e$

$$\langle \Psi, \vec\sigma_{(1)}\cdot\vec e\,\Psi\rangle = 0$$

Hence: $\langle(\vec\sigma\cdot\vec e_1)_{(1)}(\vec\sigma\cdot\vec e_2)_{(2)}\rangle = -\langle(\vec\sigma\cdot\vec e_1)_{(1)}(\vec\sigma\cdot\vec e_2)_{(1)}\rangle$

$$= -\langle(\vec e_1\cdot\vec e_2)\mathbb{1}_{(1)} + i\vec\sigma\cdot(\vec e_1\wedge\vec e_2)_{(1)}\rangle = -\vec e_1\cdot\vec e_2$$

Picking vectors:



i.e. $\vec e_1\cdot\vec e_2 = \vec e_1{}'\cdot\vec e_2 = \vec e_1\cdot\vec e_2{}' = \cos\frac{\pi}{4} = \frac{\sqrt2}{2}$

$$e_1'\cdot e_2' = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt2}{2}$$

$\Rightarrow$ (l.h.s. of (*)) $= 3\cdot\frac{\sqrt2}{2} + \frac{\sqrt2}{2} = 2\sqrt2$ indeed. $\square$