MATH 421/510, 2019WT2

Homework set 6 – Solution

Problem 1. (i) Let $w \in \ell^1$. Since any $z \in C_0$ is bounded, elementary properties of sequences yield $|\sum_{n=1}^\infty w_n z_n| \leq \|z\|_\infty \|w\|_1$, where $\|z\|_\infty = \sup\{|z_n| : n \in \mathbb{N}\}$, showing that $T_w(z) = \sum_{n=1}^\infty w_n z_n$ is a bounded linear functional on C_0. We claim that reciprocally, for any $\lambda \in C_0^*$, there is a sequence $w \in \ell^1$ such that $\lambda(z) = T_w(z)$. For any $m \in \mathbb{N}$, let $\delta^m \in C_0$ be the sequence $\delta^m_n = \delta_{m,n}$. Let w be the sequence defined by $w_n = \lambda(\delta^n)$, and define

$$\tilde{w}_n = \begin{cases} \overline{w_n}/|w_n| & \text{if } w_n \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

With these definitions, $\sum_{n=1}^N |w_n| = \sum_{n=1}^N w_n \tilde{w}_n = \sum_{n=1}^N \lambda(\delta^n)\tilde{w}_n = \lambda\Big(\sum_{n=1}^N \tilde{w}_n \delta^n\Big)$ by the linearity of λ. For any fixed $N \in \mathbb{N}$, $\sum_{n=1}^N \tilde{w}_n \delta^n$ is a linear combination of elements of C_0, hence itself in C_0, and as such each of its components has magnitude at most equal to 1. We conclude that $\sum_{n=1}^N |w_n| \leq \|\lambda\|$. The sequence $(w_n)_{n\in\mathbb{N}}$ is therefore monotone and bounded, hence convergent, namely $w \in \ell^1$. Let $T_w \in C_0^*$ be as in the first part. Clearly $T_w(\delta^n) = w_n = \lambda(\delta^n)$ and so by linearity $T_w(z) = \lambda(z)$ for any sequence with a finite number of nonzero components. But this is a dense set in C_0. For $z \in C_0$, let $z^n \to z$. Then

$$|T_w(z) - \lambda(z)| \leq |T_w(z) - T_w(z^n)| + |T_w(z^n) - \lambda(z^n)| + |\lambda(z^n) - \lambda(z)| \leq (\|T_w\| - \|\lambda\|)\|z^n - z\|_\infty,$$

proving that $T_w = \lambda$ on C_0.

(ii) We first note that the functional $L(z) = \lim_{n\to\infty} z_n$ is in C^*. Indeed, $|L(z)| = |\lim_{n\to\infty} z_n| \leq \sup\{|z_n| : n \in \mathbb{N}\} = \|z\|_\infty$. We now claim that $\lambda \in C^*$ if and only if there is $\tau \in \mathbb{C}$ and $w \in \ell^1$ such that $\lambda(z) = \tau L(z) + T_w(z)$. Let $\lambda \in C^*$. The sequences δ^n define in (i) are elements of C, and we let again $w_n = \lambda(\delta^n)$. As above, $w \in \ell^1$, and T_w defines a bounded linear functional on C^*. Let $\tau = \lambda(1) - \sum_{n=1}^\infty z_n$, where the series is convergent since it is absolutely convergent and $1 \in C$ denotes the constant sequence $1_n = 1$ for all $n \in \mathbb{N}$. But

$$(\tau L + T_w)(\delta^n) = w_n = \lambda(\delta^n) \quad \text{and} \quad (\tau L + T_w)(1) = \tau + \sum_{n=1}^\infty w_n = \lambda(1)$$

by definition of τ. By linearity, $(\tau L + T_w)(z) = \lambda$ for any z with finitely many different components, and since this is a dense subset of C, we conclude as above that $\tau L + T_w = \lambda$ on C. Finally, we note that we proved along the way that $\tau L + T_w \in C^*$ for any $\tau \in \mathbb{C}$ and $w \in \ell^1$, concluding the proof.

Problem 2. (i) If $F, G \in A^\perp$ and $\lambda \in \mathbb{C}$, then $(F + \lambda G)(x) = F(x) + \lambda G(x) = 0$ showing that $A^\perp$ is a linear subspace. Since the norm topology is metric, it is first countable and hence it suffices to prove that the limit of any convergent sequence $(F_n)_{n\in\mathbb{N}}$ in $A^\perp$ belongs to $A^\perp$. But that is immediate since $x \in A$ implies that $0 = F_n(x)$ for all $n \in \mathbb{N}$ and in turn that $F(x) = 0$, namely $F \in A^\perp$.

(ii) Let F, F' both be in $[F]$, namely there is $G \in A^\perp$ such that $F' = F + G$, and hence $F' \restriction_A = F \restriction_A + G \restriction_A = F \restriction_A$ so that J is well-defined. Note that J is linear. Now

$$J([F]) = 0 \quad \Leftrightarrow \quad F \restriction_A = 0 \quad \Leftrightarrow \quad F \in A^\perp \quad \Leftrightarrow \quad [F] = 0$$

so that J is injective. By Hahn-Banach's theorem any bounded linear functional $f \in A^*$ has a bounded linear extension $F \in X^*$, showing that J is surjective. We now recall that $\|[F]\| = \inf\{\|G\| : G \in [F]\}$ and that $X^*/A^\perp$ is complete since X^* is Banach and A is closed. For any $G \in A^\perp, F \in X^*$,

$$\|F + G\|_{X^*} = \sup\{|F(x) + G(x)|/\|x\|_X : 0 \neq x \in X\} \geq \sup\{\|F(x)\|/\|x\|_X : 0 \neq x \in A\} = \|F \restriction_A\|_{A^*}.$$

showing that $\|[F]\|_{X^*/A^\perp} \geq \|F \restriction_A \|_{A^*}$. Reciprocally, let $F \in X^*$. By Hahn-Banach, its restriction $F \restriction_A$ has an extension $G \in X^*$ such that $\|G\|_{X^*} = \|F \restriction_A \|_{A^*}$, and $F - G \in A^\perp$ namely $[F] = [G]$. Hence, $\|[F]\|_{X^*/A^\perp} \leq \|[G]\|_{X^*} = \|F \restriction_A \|_{A^*}$. Hence J is isometric.

Problem 3. (i) Linearity follows from $(T^\times(g+\lambda f))(x) = (g+\lambda f)(T(x)) = g(T(x))+\lambda f(T(x)) = (T^\times g)(x)+(T^\times(\lambda f))(x)$. In order to prove boundedness, we compute

$$\|T^\times\| = \sup\left\{\frac{\|T^\times g\|_{V^*}}{\|g\|_{W^*}} : 0 \neq g \in W^*\right\} = \sup_{x\in V}\sup_{g\in W^*}\frac{|(T^\times g)(x)|}{\|x\|_V\|g\|_{W^*}} = \sup_{x\in V}\sup_{g\in W^*}\frac{|g(T(x))|}{\|x\|_V\|g\|_{W^*}}$$

By a corollary of Hahn-Banach, $\sup\{|g(T(x))|/\|g\|_{W^*} : g \in W^*\} = \|T(x)\|_V$, and further $\sup\{\|T(x)\|_V/\|x\|_V : v \in V\} = \|T\|$ by definition.
(ii) $g \in \mathrm{Ker}(T^\times)$ implies $g(T(x)) = 0$, namely $g \in \mathrm{Ran}(T)^\perp$, proving $\mathrm{Ker}(T^\times) \subset \mathrm{Ran}(T)^\perp$. Reciprocally, if $g \in \mathrm{Ran}(T)^\perp$, then $(T^\times g)(x) = 0$ for all $x \in V$, namely $T^\times g = 0$, proving $\mathrm{Ker}(T^\times) \supset \mathrm{Ran}(T)^\perp$, and hence the claim.
(iii) $x \in \mathrm{Ker}(T)$ implies $(T^\times g)(x) = 0$ for all $g \in W^*$, namely $x \in \mathrm{Ran}(T^\times)^\perp$ showing $\mathrm{Ker}(T) \subset \mathrm{Ran}(T^\times)^\perp$. Reciprocally, if $x \in \mathrm{Ker}(T^\times g)$ for all $g \in W^*$, then $g(T(x)) = 0$ for all $g \in W^*$. Since bounded linear functionals separate points by Hahn-Banach, this implies that $T(x) = 0$, namely $x \in \mathrm{Ker}(T)$ and hence $\mathrm{Ker}(T) \supset \mathrm{Ran}(T^\times)^\perp$.
(iv) Immediate from $((ST)^\times g)(x) = g(ST(x)) = (S^\times g)(T(x)) = (T^\times S^\times g)(x)$. Using this with $S = T^{-1}$ proves the second claim.

Problem 4. We apply Hahn-Banach for vector spaces to the subspace $X = \{\alpha v + w : \alpha \in \mathbb{C}, w \in W\}$ and the functional $\lambda(\alpha v + w) = \alpha d$. This is a well-defined functional since a vector in X has a unique decomposition in $\alpha v + w$. It is linear and bounded. Indeed, if $\alpha \neq 0$, then

$$\lambda(v + w/\alpha) = d = \mathrm{dist}(v, W) \leq \|v - (-w/\alpha)\|,$$

proving that $\|\lambda\|_{X^*} \leq 1$. Hence there is $\ell \in V^*$ such that $\|\ell\|_{V^*} = \|\lambda\|_{X^*} \leq 1$, as well as $\ell(w) = \lambda(w) = 0$ for all $w \in W \subset X$ and $\ell(v) = \lambda(v) = d$ since $\{\alpha x : \alpha \in \mathbb{C}\} \subset X$.