Problem 1. The image of a compact set by a continuous function being compact, \{|f(x)| : x \in M\} is a closed and bounded subset of \([0, \infty)\) and so has a well-defined supremum which is in \([0, \infty)\). Hence \(\|\cdot\|\) is well-defined, and \(\|f\| \geq 0\). Let \(f \in C_C(M)\). Then \(\|f\| = 0\) is equivalent to \(\sup\{|f(x)| : x \in M\} = 0\), which is equivalent to \(|f(x)| = 0\) for all \(x \in M\), namely \(f = 0\). Let \(\lambda \in \mathbb{C}\). Then \(\|\lambda f\| = \sup\{|\lambda f(x)| : x \in M\} = \sup\{|\lambda ||f(x)| : x \in M\} = |\lambda| \sup\{|f(x)| : x \in M\} = \|f\|\) indeed. Finally, \(|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|\) for all \(x \in M\) implies that \(|f + g| = \sup\{|f(x) + g(x)| : x \in M\} \leq \|f\| + \|g\|\). Hence \(\|\cdot\|\) is a norm. It remains to prove that \(C_C(M)\) is complete. Let \((f_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(C_C(M)\). Since \(|f_n(x) - f_m(x)| \leq \|f_n - f_m\|\) as above, \((f_n(x))_{n \in \mathbb{N}}\) is a Cauchy sequence in \(C\) for all \(x \in M\). Let \(f(x)\) be its limit. Now,

\[
|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \sup\{|f_n(x) - f_m(x)| : m \geq n\} \leq \sup\{|f_n - f_m| : m \geq n\}
\]

vanishes as \(n \to \infty\) since \((f_n)_{n \in \mathbb{N}}\) is Cauchy. In other words, \((f_n)_{n \in \mathbb{N}}\) converges uniformly to the function \(f\), which is consequently continuous.

Problem 2. (i) Let \(W\) be complete. A normed space being a metric space, it is first countable and Hausdorff. Hence \(x \in W\) implies that there is \((x_n)_{n \in \mathbb{N}}\) in \(W\) with \(x_n \to x\). In particular, \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(W\), hence convergent, say to \(y \in W\). But limits are unique in Hausdorff spaces, so that \(x = y \in W\) and \(W\) is closed. Reciprocally, let \(W\) be closed and let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(W\). Since \(V\) is complete, \(x_n \to x \in V\). But \(x \in W\) since \(W\) is closed. It follows that any Cauchy sequence in \(W\) is convergent in \(W\), namely \(W\) is complete.

(ii) The inequality holds if \(S = \sum_{i=1}^{n} |\lambda_i| = 0\). Assume that \(S > 0\). By redefining \(\lambda_i \to \lambda_i/S\), it suffices to consider the case \(S = 1\). Assume that no such lower bound exists. Then there is a sequence \((y_m)_{m \in \mathbb{N}}\) given by \(y_m = \sum_{i=1}^{n} \lambda_{i,m}x_i\) with \(\sum_{i=1}^{n} |\lambda_{i,m}| = 1\) for all \(m \in \mathbb{N}\), and such that \(y_m \to 0\) as \(m \to \infty\). Since \(|\lambda_{i,m}| \leq 1\) for all \((i, m)\), the bounded sequence \((\lambda_{i,m})_{m \in \mathbb{N}}\) in \(\mathbb{C}\) has a convergent subsequence with limit \(\lambda_i\) and let \((y^{(1)}_m)_{m \in \mathbb{N}}\) be the corresponding subsequence of \((y_m)_{m \in \mathbb{N}}\). Repeating this with \((y^{(1)}_m)_{m \in \mathbb{N}}\) instead of \((y_m)_{m \in \mathbb{N}}\), and then recursively for a total of \(n\) times, we obtain a sequence \((y^{(n)}_m)_{m \in \mathbb{N}}\) instead of \((y^{(n)}_m)_{m \in \mathbb{N}}\), with \(\sum_{i=1}^{n} |\lambda_{i,m}| = 1\) and \(\lim_{m \to \infty} \lambda_{i,m}^{(n)} = \lambda_i\) for any \(1 \leq i \leq n\). Hence \(\lim_{m \to \infty} y^{(n)}_m = \sum_{i=1}^{n} \lambda_i x_i\). But \((y^{(n)}_m)_{m \in \mathbb{N}}\) being a subsequence of the original \((y_m)_{m \in \mathbb{N}}\), we have that \(\sum_{i=1}^{n} \lambda_i x_i = 0\) so that by linear independence \(\lambda_i = 0\) for all \(1 \leq i \leq n\), which is a contradiction.

(iii) Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(W\) and let \(\lambda_{i,n}\) be the coefficients of \(x_n\) in the basis. For \(\epsilon > 0\), there \(N \in \mathbb{N}\) such that \(k, l \geq N\) implies

\[
\epsilon > |x_k - x_l| = \|\sum_{i=1}^{n} (\lambda_{i,k} - \lambda_{i,l}) e_i\| \geq c \sum_{i=1}^{n} |\lambda_{i,k} - \lambda_{i,l}|,
\]

where \(c > 0\). Hence, for any \(1 \leq i \leq n\), the sequence \((\lambda_{i,n})_{n \in \mathbb{N}}\) is Cauchy and therefore convergent in \(\mathbb{C}\). Let \(\lambda_i\) be its limit, and let \(x = \sum_{i=1}^{n} \lambda_i e_i \in W\). But then \(|x_n - x| \leq \sum_{i=1}^{n} |\lambda_{i,k} - \lambda_i||e_i|| \to 0\) proving that \((x_n)_{n \in \mathbb{N}}\) is convergent in \(W\), which is what we had set to prove.

Remark. In (i), the proof of \((\text{complete} \Rightarrow \text{closed})\) does not use the completeness of \(V\). We conclude that any finite dimensional subspace of a normed vector space is closed.

Problem 3. (i) The metric topology being generated by open balls, it suffices to show that any open ball w.r.t. \(\|\cdot\|_1\) is contained in an open ball w.r.t. \(\|\cdot\|_2\) and vice versa. But that is exactly the property defining equivalent norms.

(ii) Let \(\{e_1, \ldots, e_n\}\) be a basis of \(V\) and let \(K = \mathbb{C}\) or \(K = \mathbb{R}\). Let \(i : K^n \to V\) be the linear bijection defined by \(i(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i e_i\). Furthermore, let \(\|\cdot\|_2 : V \to [0, \infty)\) be defined by \(\|\sum_{i=1}^{n} \lambda_i e_i\|_2 = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{1/2}\),
namely \( \|i(A)\|_2 = \|A\| \), where \( \| \cdot \| \) is the standard Pythagorean norm in \( \mathbb{K}^n \). Hence \( \| \cdot \|_2 \) is a norm on \( V \) and \( i \) is a bounded bijection from \( (\mathbb{K}^n, \| \cdot \|) \) to \( (V, \| \cdot \|_2) \). It now suffices to prove that any other norm \( \| \cdot \|_1 \) is equivalent to \( \| \cdot \|_2 \). Let \( \ell : V \to [0, \infty) \) (with \( V \) equipped with \( \| \cdot \|_2 \)) be defined by \( \ell(v) = \|v\|_1 \). Then 
\[
\|v\|_1 \leq \sum_{i=1}^{n} |\langle \lambda_i \rangle|e_i \leq n \max\{\|e_i\|_1 : 1 \leq i \leq n\} \max\{|\lambda_i| : 1 \leq i \leq n\} \leq C(\sum_{i=1}^{n} |\lambda_i|^2)^{1/2} = C\|v\|_2,
\]
where \( C = n \max\{|\lambda_i| : 1 \leq i \leq n\} \), proving that \( \ell \) is a continuous map. Now the image \( S \subset V \) by \( i \) of the unit sphere is the image of a compact set by a continuous function, hence it is itself compact. It follows that \( \ell(S) \) is compact and hence there are \( m, M \in [0, \infty) \) such that \( m \leq \ell(v) \leq M \) for all \( v \in S \). Since \( \|v\|_2 \neq 0 \) for all nonzero \( v \in V \), we conclude that \( m \leq \|v\|_1 \leq M \) for all nonzero \( v \in V \), which is the desired equivalence after multiplication by \( \|v\|_2 \).

**Problem 4.** (i) We first note that \([v] + [w] = [v + w] \) and \([\lambda v] = \lambda[v] \) are well-defined since \( C \) is a vector subspace, namely closed under addition and scalar multiplication. Hence, \( V/C \) is a vector space. We check the three axioms of the norm.

\[
\|\lambda[v]\| = \|\lambda[v]\| = \inf\{\|\lambda[v] + w\| : w \in C\} = \inf\{\|\lambda v + w\| : w \in C\} = \|\lambda\|\inf\{\|w\| : w \in [v]\} = \|\lambda\|\|v\|
\]

\[
\|[v + w]\| = \inf\{\|v + w + 2z\| : z \in C\} \leq \inf\{\|v + z\| + \|w + z\| : z \in C\} \leq \|v\| + \|w\|
\]

while \( \|v\| \geq 0 \), being the infimum over a set of non-negative numbers. It remains to prove the non-degenerate property. If \( \|v\| = 0 \), there is a sequence \((w_n)_{n \in \mathbb{N}} \) in \( C \) such that \( \lim_{n \to \infty} \|v + w_n\| = 0 \), namely \((w_n)_{n \in \mathbb{N}} \) converges to \( v \) in \( V \). Since \( C \) is closed, we must have \( v \in C \), namely \([v] = 0\).

(ii) Let \( \sum_{n=1}^{\infty} \|[v_n]\| \) be convergent. For any \( n \in \mathbb{N} \), there is \( w_n \sim v_n \) such that \( \|w_n\| \leq \|[v_n]\| + 2^{-n} \). Hence \( \sum_{n=1}^{\infty} \|[v_n]\| \) is convergent and so is \( \sum_{n=1}^{\infty} w_n = w \) since \( V \) is complete. Hence,

\[
\|[w] - \sum_{n=1}^{N} [v_n]\| = \|[w] - \sum_{n=1}^{N} [w_n]\| \leq \|w - \sum_{n=1}^{N} w_n\|
\]

vanishes as \( n \to \infty \), proving that \( \sum_{n=1}^{\infty} [v_n] \) is convergent. Hence \( V/C \) is complete.

**Problem 5.** (i) Let us first consider the case \( q \leq \infty \). We write \( r = \lambda p + (1 - \lambda)q \), where \( \lambda = \frac{q}{p} \) and apply Hölder’s inequality to \( |f|^{\lambda p} \) and \( |f|^{(1-\lambda)q} \) with indices \( 1/\lambda, 1/(1-\lambda) \) to get \( \|f\|_r^p \leq \|f\|_p^p/\|f\|_q^{(1-\lambda)q} \). If \( q = \infty \), we similarly use Hölder’s inequality with \( |f|^p \) and \( |f|^{-p} \) and indices \( 1, \infty \) to get \( \|f\|_r^p \leq \|f\|_p^p/\|f\|^{-p}_\infty = \|f\|_p^p/\|f\|^{1-p}_\infty \).

(ii) By definition, the set \( A_t = \{x \in \Omega : |f(x)| \geq t\} \) has positive measure for all \( t < \|f\|_\infty \), and

\[
\|f\|_q \geq \left( \int_{A_t} |f|^q d\mu \right)^{1/q} \geq t \mu(A_t)^{1/q}
\]

for any \( 1 \leq q < \infty \). Hence, \( \liminf_{q \to \infty} \|f\|_q \geq t \) and since this holds for any \( t < \|f\|_\infty \), we conclude that \( \liminf_{q \to \infty} \|f\|_q \geq \|f\|_\infty \). Now, since \( f \in L^p(\Omega) \), the last part of (i) yields \( \|f\|_q \leq \|f\|_p^p/\|f\|_\infty^{1-p/q} \) for any \( q > p \) and hence \( \limsup_{q \to \infty} \|f\|_q \leq \|f\|_\infty \).