MATH 421/510, 2019WT2

Homework set 4 – Solution

Problem 1. The image of a compact set by a continuous function being compact, $\{|f(x)| : x \in M\}$ is a closed and bounded subset of $[0, \infty)$ and so has a well-defined supremum which is in $[0, \infty)$. Hence $\|\cdot\|$ is well-defined, and $\|f\| \geq 0$. Let $f \in C_{\mathbb{C}}(M)$. Then $\|f\| = 0$ is equivalent to $\sup\{|f(x)| : x \in M\} = 0$, which is equivalent to $|f(x)| = 0$ for all $x \in M$, namely $f = 0$. Let $\lambda \in \mathbb{C}$. Then $\|\lambda f\| = \sup\{|\lambda f(x)| : x \in M\} = \sup\{|\lambda||f(x)| : x \in M\} = |\lambda| \sup\{|f(x)| : x \in M\} = |\lambda|\|f\|$ indeed. Finally, $|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \|f\|+\|g\|$ for all $x \in M$ implies that $\|f+g\| = \sup\{|f(x)+g(x)| : x \in M\} \leq \|f\|+\|g\|$. Hence $\|\cdot\|$ is a norm. It remains to prove that $C_{\mathbb{C}}(M)$ is complete. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C_{\mathbb{C}}(M)$. Since $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$ as above, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $x \in M$. Let $f(x)$ be its limit. Now,

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \sup\{|f_n(x) - f_m(x)| : m \geq n\} \leq \sup\{\|f_n - f_m\| : m \geq n\}$$

vanishes as $n \to \infty$ since $(f_n)_{n\in\mathbb{N}}$ is Cauchy. In other words, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function f, which is consequently continuous.

Problem 2. (i) Let W be complete. A normed space being a metric space, it is first countable and Hausdorff. Hence $x \in \overline{W}$ implies that there is $(x_n)_{n\in\mathbb{N}}$ in W with $x_n \to x$. In particular, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in W, hence convergent, say to $y \in W$. But limits are unique in Hausdorff spaces, so that $x = y \in W$ and W is closed. Reciprocally, let W be closed and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in W. Since V is complete, $x_n \to x \in V$. But $x \in W$ since W is closed. It follows that any Cauchy sequence in W is convergent in W, namely W is complete.
(ii) The inequality holds if $S = \sum_{i=1}^n |\lambda_i| = 0$. Assume that $S > 0$. By redefining $\lambda_i \to \lambda_i/S$, it suffices to consider the case $S = 1$. Assume that no such lower bound exists. Then there is a sequence $(y_m)_{m\in\mathbb{N}}$ given by $y_m = \sum_{i=1}^n \lambda_{i,m} x_i$ with $\sum_{i=1}^n |\lambda_{i,m}| = 1$ for all $m \in \mathbb{N}$, and such that $y_m \to 0$ as $m \to \infty$. Since $|\lambda_{i,m}| \leq 1$ for all (i, m), the bounded sequence $(\lambda_{1,m})_{m\in\mathbb{N}}$ in $\mathbb{C}$ has a convergent subsequence with limit λ_1 and let $(y_m^{(1)})_{m\in\mathbb{N}}$ be the corresponding subsequence of $(y_m)_{m\in\mathbb{N}}$. Repeating this with $(y_m^{(1)})_{m\in\mathbb{N}}$ instead of $(y_m)_{m\in\mathbb{N}}$, and then recursively for a total of n times, we obtain a sequence $(y_m^{(n)})_{m\in\mathbb{N}}$ with $y_m^{(n)} = \sum_{i=1}^n \lambda_{i,m}^{(n)} x_i$ for which $\sum_{i=1}^n |\lambda_{i,m}^{(n)}| = 1$ and $\lim_{m\to\infty} \lambda_{i,m}^{(n)} = \lambda_i$ for any $1 \leq i \leq n$. Hence $\lim_{m\to\infty} y_m^{(n)} = \sum_{i=1}^n \lambda_i x_i$. But $(y_m^{(n)})_{m\in\mathbb{N}}$ being a subsequence of the original $(y_m)_{m\in\mathbb{N}}$, we have that $\sum_{i=1}^n \lambda_i x_i = 0$ so that by linear independence $\lambda_i = 0$ for all $1 \leq i \leq n$, which is a contradiction.
(iii) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in W and let $\lambda_{i,n}$ be the coefficients of x_n in the basis. For $\epsilon > 0$, there $N \in \mathbb{N}$ such that $k, l \geq N$ implies

$$\epsilon > \|x_k - x_l\| = \|\sum_{i=1}^n (\lambda_{i,k} - \lambda_{i,l}) e_i\| \geq c \sum_{i=1}^n |\lambda_{i,k} - \lambda_{i,l}|,$$

where $c > 0$. Hence, for any $1 \leq i \leq n$, the sequence $(\lambda_{i,n})_{n\in\mathbb{N}}$ is Cauchy and therefore convergent in $\mathbb{C}$. Let λ_i be its limit, and let $x = \sum_{i=1}^n \lambda_i e_i \in W$. But then $\|x_n - x\| \leq \sum_{i=1}^n |\lambda_{i,k} - \lambda_i|\|e_i\| \to 0$ proving that $(x_n)_{n\in\mathbb{N}}$ is convergent in W, which is what we had set to prove.
Remark. In (i), the proof of (complete $\Rightarrow$ closed) does not use the completeness of V. We conclude that any finite dimensional subspace of a normed vector space is closed.

Problem 3. (i) The metric topology being generated by open balls, it suffice to show that any open ball w.r.t. $\|\cdot\|_1$ is contained in an open ball w.r.t. $\|\cdot\|_2$ and vice versa. But that is exactly the property defining equivalent norms.
(ii) Let $\{e_1, \ldots, e_n\}$ be a basis of V and let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Let $i : \mathbb{K}^n \to V$ be the linear bijection defined by $i(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i e_i$. Furthermore, let $\|\cdot\|_2 : V \to [0, \infty)$ be defined by $\|\sum_{i=1}^n \lambda_i e_i\|_2 = (\sum_{i=1}^n |\lambda_i|^2)^{1/2}$,

namely $\|i(\underline{\lambda})\|_2 = \|\underline{\lambda}\|$, where $\|\cdot\|$ is the standard Pythagorean norm in $\mathbb{K}^n$. Hence $\|\cdot\|_2$ is a norm on V and i is a bounded bijection from $(\mathbb{K}^n, \|\cdot\|)$ to $(V, \|\cdot\|_2)$. It now suffices to prove that any other norm $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$. Let $\ell : V \to [0,\infty)$ (with V equipped with $\|\cdot\|_2$) be defined by $\ell(v) = \|v\|_1$. Then $v = \sum_{i=1}^n \lambda_i e_i$ and

$$\|v\|_1 \leq \sum_{i=1}^n |\lambda_i| \|e_i\|_1 \leq n \max\{\|e_i\|_1 : 1 \leq i \leq n\} \max\{|\lambda_i| : 1 \leq i \leq n\} \leq C(\sum_{i=1}^n |\lambda_i|^2)^{1/2} = C\|v\|_2,$$

where $C = n\max\{\|e_i\|_1 : 1 \leq i \leq n\}$, proving that ℓ is a continuous map. Now the image $S \subset V$ by i of the unit sphere is the image of a compact set by a continuous function, hence it is itself compact. It follows that $\ell(S)$ is compact and hence there are $m, M \in [0,\infty)$ such that $m \leq \ell(v) \leq M$ for all $v \in S$. Since $\|v\|_2 \neq 0$ for all nonzero $v \in V$, we conclude that $m \leq \|v/\|v\|_2\|_1 \leq M$ for all nonzero $v \in V$, which is the desired equivalence after multiplication by $\|v\|_2$.

Problem 4. (i) We first note that $[v] + [w] = [v + w]$ and $[\lambda v] = \lambda[v]$ are well-defined since C is a vector subspace, namely closed under addition and scalar multiplication. Hence, V/C is a vector space. We check the three axioms of the norm.

$$\|\lambda[v]\| = \|[\lambda v]\| = \inf\{\|\lambda v + w\| : w \in C\} = \inf\{|\lambda|\|v + w\| : w \in C\} = |\lambda| \inf\{\|w\| : w \in [v]\} = |\lambda|\|[v]\|$$
$$\|[v + w]\| = \inf\{\|v + w + 2z\| : z \in C\} \leq \inf\{\|v + z\| + \|w + z\| : z \in C\} \leq \|[v]\| + \|[w]\|$$

while $\|[v]\| \geq 0$, being the infimum over a set of non-negative numbers. It remains to prove the non-degenerate property. If $\|[v]\| = 0$, there is a sequence $(w_n)_{n\in\mathbb{N}}$ in C such that $\lim_{n\to\infty} \|v + w_n\| = 0$, namely $(-w_n)_{n\in\mathbb{N}}$ converges to v in V. Since C is closed, we must have $v \in C$, namely $[v] = 0$.
(ii) Let $\sum_{n=1}^\infty \|[v_n]\|$ be convergent. For any $n \in \mathbb{N}$, there is $w_n \sim v_n$ such that $\|w_n\| \leq \|[v_n]\| + 2^{-n}$. Hence $\sum_{n=1}^\infty \|w_n\|$ is convergent and so is $\sum_{n=1}^\infty w_n = w$ since V is complete. Hence,

$$\|[w] - \sum_{n=1}^N [v_n]\| = \|[w] - \sum_{n=1}^N [w_n]\| = \|[w - \sum_{n=1}^N w_n]\| \leq \|w - \sum_{n=1}^N w_n\|$$

vanishes as $n \to \infty$, proving that $\sum_{n=1}^\infty [v_n]$ is convergent. Hence V/C is complete.

Problem 5. (i) Let us first consider the case $q < \infty$. We write $r = \lambda p + (1-\lambda)q$, where $\lambda = \frac{r-q}{p-q}$ and apply Hölder's inequality to $|f|^{\lambda p}$ and $|f|^{(1-\lambda)q}$ with indices $1/\lambda, 1/(1-\lambda)$ to get $\|f\|_r^r \leq \|f\|_p^{\lambda/q}\|f\|_q^{(1-\lambda)/q}$. If $q = \infty$, we similarly use Hölder's inequality with $|f|^p$ and $|f|^{r-p}$ and indices $1, \infty$ to get $\|f\|_r^r \leq \|f\|_p^p \||f|^{r-p}\|_\infty = \|f\|_p^p \|f\|_\infty^{r-p}$.
(ii) By definition, the set $A_t = \{x \in \Omega : |f(x)| \geq t\}$ has positive measure for all $t < \|f\|_\infty$, and

$$\|f\|_q \geq \Big(\int_{A_t} |f|^q d\mu\Big)^{1/q} \geq t\mu(A_t)^{1/q}$$

for any $1 \leq q < \infty$. Hence, $\liminf_{q\to\infty} \|f\|_q \geq t$ and since this holds for any $t < \|f\|_\infty$, we conclude that $\liminf_{q\to\infty} \|f\|_q \geq \|f\|_\infty$. Now, since $f \in L^p(\Omega)$, the last part of (i) yields $\|f\|_q \leq \|f\|_p^{p/q}\|f\|_\infty^{1-p/q}$ for any $q > p$ and hence $\limsup_{q\to\infty} \|f\|_q \leq \|f\|_\infty$.