MATH 421/510, 2019WT2

Homework set 11 – Solution

Problem 1. Denote $a_1 = \sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |A(n,m)|$ and $a_2 = \sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} |A(n,m)|$ Then, by the Cauchy-Schwarz inequality, for each fixed $n \in \mathbb{N}$,

$$|(Ab)_n| \leq \sum_{m\in\mathbb{N}} |A(n,m)b_m| \leq \sqrt{a_1}\Big(\sum_{m\in\mathbb{N}} |A(n,m)||b_m|^2\Big)^{1/2}$$

Hence

$$\|Ab\|^2 = \sum_{n\in\mathbb{N}} |(Ab)_n|^2 \leq a_1 \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |A(n,m)||b_m|^2 \leq a_1 a_2 \sum_{m\in\mathbb{N}} |b_m|^2 = a_1 a_2 \|b\|^2$$

proving that A is a bounded operator on ℓ^2 with $\|A\| \leq \sqrt{a_1 a_2}$.

Problem 2. (i) '$\Rightarrow$', by contradiction: We assume that there is $\epsilon_0 > 0$ such that $\mu\{x \in X : |f(x) - \lambda| < \epsilon_0\} = 0$. Then $|f(x) - \lambda| \geq \epsilon_0$ for μ almost all $x \in X$ so that the operator of multiplication by $(f(x) - \lambda)^{-1}$ is bounded by ϵ_0^{-1}. Clearly, this is also the inverse of T_f. Hence $\lambda \notin \sigma(T_f)$.
'$\Leftarrow$', by contradiction: We assume that $\lambda \notin \sigma(T_f)$. Then $T_f - \lambda 1$ is invertible with bounded inverse and we denote $M = \|(T_f - \lambda 1)^{-1}\|$. Let S be a set of finite positive measure such that $|f(x) - \lambda| < (2M)^{-1}$ for every $x \in S$. Its characteristic function has finite L^2 norm, namely $\|\chi_S\|_2^2 = \mu(S)$. But then

$$\|(T_f - \lambda 1)\chi_S\|_2 \leq (2M)^{-1}\sqrt{\mu(S)} = (2M)^{-1}\|\chi_S\|_2.$$

Let now $g = (T_f - \lambda 1)\chi_S$. It follows that

$$\|(T_f - \lambda 1)^{-1} g\|_2 = \|\chi_s\|_2 \geq 2M\|(T_f - \lambda 1)\chi_S\|_2 = 2M\|g\|_2$$

which is a contradiction with the definition of M.
(ii) '$\Rightarrow$': Let $0 \neq g_\lambda \in \mathcal{H}$ be such that $T_f g_\lambda = \lambda g_\lambda$. This implies that $(f(x) - \lambda)g_\lambda(x) = 0$ for μ almost every $x \in X$. Assume now that $\mu\{x \in X : |f(x) = \lambda\} = 0$. Then $g_\lambda(x) = 0$ for μ almost every $x \in X$, namely $g_\lambda = 0$ as an L^2 function, contradiction.
'$\Leftarrow$': Let S be a set of finite positive measure such that $f(x) = \lambda$ for every $x \in S$. Then χ_S is a non-zero L^2 function, but $(f(x) - \lambda)\chi_S(x) = 0$ for all $s \in S$, namely $\lambda \in \sigma(T_f)$.
(iii) Since $\{x \in (0,1) : x = \lambda\}$ is either $\{\lambda\}$ if $\lambda \in (0,1)$ or $\emptyset$ otherwise, the conclude by (ii) that the multiplication operator by x has no eigenvalue. However, if $\lambda \in [0,1]$, then for $\epsilon > 0$, $\{x \in (0,1) : |x\lambda| < \epsilon\} = (\lambda - \epsilon, \lambda + \epsilon)$, which has finite measure. If $\lambda \notin (0,1)$, the measure of the corresponding set vanishes for ϵ small enough. Hence $\sigma(T_f) = [0,1]$.

Problem 3. (i) First, we check that $\{\varphi_n : n \in \mathbb{Z}\}$ is an orthonormal set in $L^2([0,2\pi])$. If m and n are distinct integers,

$$\langle \varphi_n, \varphi_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{i(n-m)\theta}\, d\theta = \frac{1}{2\pi i(n-m)} \mathrm{e}^{\mathrm{i}(n-m)\theta}\Big|_0^{2\pi} = 0$$

since $\mathrm{e}^{\mathrm{i}t}$ is periodic of period 2π. Thus, when m and n are distinct integers, φ_n and φ_m are orthogonal. If n is any integer,

$$\langle \varphi_n, \varphi_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-n)\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1.$$

Thus, when n is an integer, φ_n is normalized.
Hence by Bessel's inequality $\sum_{j=n}^m |c_j|^2 \leq \|f\|_2^2 < \infty$ for any integers $n < m$. This implies $\lim_{n\to\infty} c_{\pm n} = 0$.

(ii) We compute

$$S_N(\theta) = \frac{1}{2\pi}\sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n\theta}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n(\theta-x)}\, dx$$

Now the sum is the difference of two partial sums of geometric series, namely

$$\sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n(\theta-x)} = \frac{\mathrm{e}^{-\mathrm{i}N(\theta-x)} - \mathrm{e}^{\mathrm{i}(N+1)(\theta-x)}}{1-\mathrm{e}^{\mathrm{i}(\theta-x)}} = \frac{\sin((N+1/2)(\theta-x))}{\sin\frac{\theta-x}{2}}$$

It remains to plug this in the expression above, make the change of variables $x \to x+\theta$, and use the periodicity of the integrand to restore the limits of integration.

Problem 4. (i) First of all, we note that by (i) of Problem 3,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}dx = \sum_{n=-N}^{N}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nx}dx = 1,$$

so that using (ii) of Problem 3,

$$S_N(\theta) - f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} (f(x+\theta)-f(\theta))\frac{\sin((N+1/2)x)}{\sin(x/2)}dx = \langle\varphi_{-N}, g\rangle - \langle\varphi_N, h\rangle$$

where φ_N were defined in Problem 3, and

$$g(x) = \mathrm{e}^{\mathrm{i}x/2}\frac{f(x+\theta)-f(\theta)}{2\sqrt{2\pi}\mathrm{i}\sin(x/2)}, \qquad h(x) = \mathrm{e}^{-\mathrm{i}x/2}\frac{f(x+\theta)-f(\theta)}{2\sqrt{2\pi}\mathrm{i}\sin(x/2)}.$$

Since g, h are continuous periodic functions, they belong to $L^2([0,2\pi])$, and we conclude by Problem 3(i) that $\lim_{N\to\infty}\langle\varphi_{-N}, g\rangle = 0 = \lim_{N\to\infty}\langle\varphi_N, h\rangle$, which yields the claim. Continuity follows from the fact that $\lim_{x\to 0}\frac{\sin(x/2)}{x/2} = 1$ and the observation that $\frac{f(x+\theta)-f(\theta)}{x} = \int_0^1 f'(\theta+tx)dt$, since the right hand side it continuous by assumption.

(ii) Convergence of $\sum|b_n|^2$ is again by Bessel's inequality with $\|f'\|_2^2 < \infty$. Moreover,

$$b_n = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nx}f'(x)dx = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{i}n\mathrm{e}^{-\mathrm{i}nx}f(x)dx = \mathrm{i}nc_n$$

where we integrated by parts and used periodicity to cancel out the boundary terms. This proves the second claim.

(iii) By Hölder's inequality, $\sum_{n\in\mathbb{Z}}|c_n| = |c_0| + \sum_{n\neq 0}\frac{1}{n}|nc_n| \le |c_0| + \left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n\neq 0} n^2|c_n|^2\right)^{1/2}$, which is convergent.

(iv) By the above, $S_N(\theta) = \sum_{n\in\mathbb{Z}} c_n\varphi_n(\theta)$ converges uniformly. By (i) the limit is $f(\theta)$ indeed.