MATH 421/510, 2019WT2

Homework set 10 – Solution

Problem 1. (i) Let $\eth\zeta = (2\pi\mathrm{i})^{-1}d\zeta$. Since

$$T^k = \oint_\gamma \zeta^k(\zeta 1 - T)^{-1}\eth\zeta,$$

we conclude that indeed

$$\oint_\gamma (\zeta 1 - T)^{-1}P(\zeta)\eth\zeta = \sum_{j=1}^N a_j \oint_\gamma (\zeta 1 - T)^{-1}\zeta^j\eth\zeta = \sum_{j=1}^N a_j T^j.$$

(ii) Linearity of $f \mapsto f(T)$ is immediate. We claim that if f, g are analytic in Ω, then $(fg)(T) = f(T)g(T)$. Let Θ, Γ be two contours as above such that Θ is completely in the interior of Γ. Then

$$f(T)g(T) = \oint_\Theta\oint_\Gamma (\theta - T)^{-1}(\gamma - T)^{-1}f(\theta)g(\gamma)\eth\gamma\eth\theta.$$

Now $(\theta - T) - (\gamma - T) = \theta - \gamma$, which yields when multiplied by $(\theta - T)^{-1}(\gamma - T)^{-1}(\theta-\gamma)^{-1}$ the identity

$$(\theta-\gamma)^{-1}\left((\gamma - T)^{-1} - (\theta - T)^{-1}\right) = (\theta - T)^{-1}(\gamma - T)^{-1}.$$

Hence

$$f(T)g(T) = \oint_\Theta\oint_\Gamma \left((\gamma - T)^{-1} - (\theta - T)^{-1}\right)\frac{f(\theta)g(\gamma)}{\theta - \gamma}\eth\gamma\eth\theta.$$

The first term vanishes since $\oint_\Theta \frac{f(\theta)}{\theta-\gamma}d\theta = 0$ because Γ lies outside of the interior of Θ. The second term reduces to

$$f(T)g(T) = -\oint_\Theta\left(\oint_\Gamma \frac{g(\gamma)}{\theta-\gamma}\eth\gamma\right)(\theta - T)^{-1}f(\theta)\eth\theta = -\oint_\Theta (\theta - T)^{-1}f(\theta)g(\theta)\eth\theta = (fg)(T).$$

(iii) Let $\mu \in f(\sigma(T)$, namely $\mu = f(\lambda)$ for some $\lambda \in \sigma(T)$. Since f is analytic, the function $F(\zeta) = (\zeta-\lambda)^{-1}(f(\zeta) - f(\lambda))$ is analytic in Ω so that $F(T)$ is a well-defined element of $\mathcal{L}(V)$. By (ii) applied to $(\zeta - \lambda)F(\zeta)$, we conclude that $(T-\lambda)F(T) = f(T) - \mu$. But $\lambda \in \sigma(T)$ implies that the left hand side is not invertible, and hence $f(T) - \mu$ is not invertible, namely $\mu \in \sigma(f(T))$. Hence $f(\sigma(T)) \subset \sigma(f(T))$. Let now $\mu \notin f(\sigma(T))$, namely $f(\lambda) - \mu \neq 0$ for all $\lambda \in \sigma(T)$. Then $g(\lambda) = (f(\lambda) - \mu)^{-1}$ is analytic in an open neighbourhood of $\sigma(T)$, and hence $g(T)$ is well-defined in $\mathcal{L}(V)$. But then $(f(\lambda) - \mu)g(\lambda) = 1$ implies by (ii) that $(f(T) - \mu)g(T) = 1$, proving that $g(T)$ is the inverse of $f(T) - \mu$, and hence $\mu \notin \sigma(f(T))$. This shows that $\sigma(f(T)) \subset f(\sigma(T))$ and concludes the proof.

Problem 2. (i) A calculation:

$$\|u+v\|^2 + \|u-v\|^2 = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle + \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle = 2\|u\|^2 + 2\|v\|^2.$$

(ii) First fo all, we check that $v \mapsto -v$ exchanges the two real terms with each other, and similarly with the imaginary terms, so that $\langle u, -v\rangle = -\langle u,v\rangle$. Let now $\alpha = \frac{m}{n}$ with $m \in \mathbb{N}\cup\{0\}, n \in \mathbb{N}$. If $m = 0$, then $\langle u, 0v\rangle = 0$ indeed. Let now $m \in \mathbb{N}$. In order to prove that $\langle u, \frac{m}{n}v\rangle = \frac{m}{n}\langle u,v\rangle$, we show equivalently with $\tilde{v} = \frac{1}{n}v$ that $n\langle u, m\tilde{v}\rangle = m\langle u, n\tilde{v}\rangle$. Since $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$, we have by induction that

$\langle u, Nv\rangle = N\langle u,v\rangle$ for any $N\in\mathbb{N}$. Hence, $n\langle u, m\tilde v\rangle = nm\langle u,\tilde v\rangle = m\langle u, n\tilde v\rangle$ indeed. We conclude that for any $\alpha = \pm\frac{m}{n}\pm \mathrm{i}\frac{p}{q}$, with $m,n,p,q\in\mathbb{N}$,

$$\langle u,\alpha v\rangle = \pm\langle u, \frac{m}{n}v\rangle \pm \langle u, \mathrm{i}\frac{p}{q}v\rangle = \pm\frac{m}{n}\langle u,v\rangle \pm \mathrm{i}\langle u,\frac{p}{q}v\rangle = \alpha\langle u,v\rangle.$$

The fact that $\langle u,v\rangle = \overline{\langle v,u\rangle}$ implies that $\langle u,u\rangle$ is real. Therefore, the last two terms vanish and hence $4\langle u,u\rangle = \|2u\|^2$, namely $\langle u,u\rangle = \|u\|^2$. This shows that $\langle u,u\rangle\geq 0$ and it vanishes iff $u=0$.
We turn to the Cauchy-Schwarz inequality. It is trivially satisfied if $\langle u,v\rangle = 0$. Hence we assume that $\langle u,v\rangle = 0$, which implies in particular that both $\|u\|\neq 0, \|v\|\neq 0$. For any $\lambda,\nu\in\mathbb{C}$ with rational real and imaginary parts, we have by the above

$$0\leq \|\lambda u+\mu v\|^2 = \langle \lambda u+\mu v, \lambda u+\mu v\rangle = |\lambda|^2\|u\|^2 + \lambda\overline{\mu}\overline{\langle u,v\rangle} + \overline{\lambda}\mu\langle u,v\rangle + |\mu|^2\|v\|^2.$$

As a function of the real and imaginary parts of λ,μ, the right hand side is continuous. It is nonnegative on a dense set of $\mathbb{C}\times\mathbb{C}$, and therefore extends by continuity to a nonnegative function on $\mathbb{C}\times\mathbb{C}$. But then, the choices $\lambda = \left(\frac{\|v\|}{\|u\|}\right)^{1/2}$ and $\overline{\mu} = -\frac{\langle u,v\rangle}{|\langle u,v\rangle|}\left(\frac{\|u\|}{\|v\|}\right)^{1/2}$ yields the Cauchy-Schwarz inequality.
Finally, let $\alpha\in\mathbb{C}$, and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with rational real and imaginary parts, and converging to α. Then

$$|\langle u,\alpha_n v\rangle - \langle u,\alpha v\rangle| = |\langle u,(\alpha_n-\alpha)v\rangle| \leq \|u\|\|(\alpha_n-\alpha)v\| = |\alpha_n-\alpha|\|u\|\|v\|$$

converges to zero and hence $\langle u,\alpha v\rangle = \lim_{n\to\infty}\langle u,\alpha_n v\rangle = \lim_{n\to\infty}\alpha_n\langle u,v\rangle = \alpha\langle u,v\rangle$, concluding the proof.

Problem 3. (i) Since $\|w\| = \sup\{|\langle u,w\rangle| : \|u\|=1\}$,

$$\|A^*\| = \sup_{v\in\mathcal{H},\|v\|=1}\|A^*v\| = \sup_{v\in\mathcal{H},\|v\|=1}\ \sup_{u\in\mathcal{H},\|u\|=1}|\langle u,A^*v\rangle| = \|A\|$$

where we noted that $|\langle u,A^*v\rangle| = |\langle Au,v\rangle| = |\langle v,Au\rangle|$.
(ii) The equality for A^* follows from $1 = 1^*$, the fact that $(TS)^* = S^*T^*$, and

$$T\in\mathrm{Gl}(\mathcal{H}) \Leftrightarrow \exists S\in\mathcal{L}(\mathcal{H}) \text{ s.t. } ST=TS=1 \Leftrightarrow \exists S\in\mathcal{L}(\mathcal{H}) \text{ s.t. } T^*S^* = S^*T^* = 1 \Leftrightarrow T^*\in\mathrm{Gl}(\mathcal{H}),$$

applied to $T = \lambda 1 - A$. As for A^{-1}, it suffices to write $\lambda 1 - A = \lambda A(A^{-1}-\lambda^{-1}1)$ to conclude that if $A\in\mathrm{Gl}(\mathcal{H})$, then $\lambda 1 - A\in\mathrm{Gl}(\mathcal{H})$ iff $(\lambda^{-1}1 - A^{-1})\in\mathrm{Gl}(\mathcal{H})$.
(iii) By Cauchy-Schwarz, $|\langle v,Av\rangle|\leq\|v\|\|Av\|\leq\|v\|^2\|A\|$, proving that $\|A\|\geq\sup\{|\langle v,Av\rangle|/\|v\|^2, v\in\mathcal{H}\}$. For the reverse inequality, we pick $v\in\mathcal{H}$ such that $|v\|=1$ and $Av\neq 0$ (if no such vector exists, then $A=0$ and there is nothing to prove). Let $u = \|Av\|^{1/2}v$ and $w = \|Av\|^{-1/2}Av$. Then $\|u\|^2 = \|Av\| = \|w\|^2$. For $x = u+w, y = u-w$, we compute

$$\langle x,Ax\rangle - \langle y,Ay\rangle = 2\langle u,Aw\rangle + 2\langle w,Au\rangle = 2\langle v,A^2v\rangle + 2\langle Av,Av\rangle = 4\|Av\|^2$$

since $A = A^*$. On the other hand, if $S = \sup\{|\langle v,Av\rangle| : \|v\|=1\}$, the triangle inequality yields that

$$|\langle x,Ax\rangle - \langle y,Ay\rangle| \leq S(\|x\|^2+\|y\|^2) \leq 2S(\|u\|^2+\|v\|^2) = 4S\|Av\|$$

by the parallelogram identity. Hence $\|Av\|\leq S$. Taking the supremum over v yields the claim.
(iv) Since A^*A is self-adjoint, we see that $\|A^*A\| = \sup\{\langle v,A^*Av\rangle : \|v\|=1\} = \sup\{\|Av\|^2 : \|v\|=1\} = \|A\|^2$. With that, and if A is normal,

$$\|A^{2^n}\|^2 = \|(A^*)^{2^n}A^{2^n}\| = \|(A^*A)^{2^n}\| = \|(A^*A)^{2^{n-1}}(A^*A)^{2^{n-1}}\| = \|(A^*A)^{2^{n-1}}\|^2$$

which yields $\|A^{2^n}\|^2 = \|A^*A\|^{2^n}$ by repeating the last steps and finally $\|A^{2^n}\|^2 = \|A\|^{2^{n+1}}$. The claim follows from this by the very definition of $r(A)$.

(v) If $\lambda \in \rho(BA)$, and $\lambda \neq 0$, then

$$(\lambda 1 - AB)(1 + A(\lambda 1 - BA)^{-1}B) = \lambda 1$$

since $\lambda((\lambda 1 - BA)^{-1} - \lambda^{-1}) = BA(\lambda 1 - BA)^{-1}$, which shows that $\lambda 1 - AB$ is invertible on the right, showing that $\sigma(BA) \setminus \{0\} \subset \sigma(AB) \setminus \{0\}$. The inverse inclusion follows by exchanging the roles of A, B.

Problem 4. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(Y,\nu)$ and let $M = \sup\{\|f_n\|_{L^2(Y,\nu)} : n \in \mathbb{N}\}$. Since, by the Riesz lemma, any Hilbert space is reflexive, there is a weakly convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$. We claim that $(Kf_{n_k})_{k\in\mathbb{N}}$ converges to Kf in the norm topology of $L^2(X,\mu)$. Since $\int_{X\times Y} |k(x,y)|^2 d\mu(x) d\nu(y) < \infty$, we have that $\int_Y |k(x,y)|^2 d\nu(y) < \infty$ for μ-almost every $x \in X$. Therefore, for any such x,

$$\lim_{k\to\infty} (Kf_{n_k})(x) = \lim_{k\to\infty} \langle \overline{k(x,\cdot)}, f_{n_k}\rangle_{L^2(Y,\nu)} = \langle \overline{k(x,\cdot)}, f\rangle_{L^2(Y,\nu)} = (Kf)(x)$$

by weak convergence. Moreover, the Cauchy-Schwarz inequality yields

$$|(Kf_{n_k})(x)| \leq \|f_{n_k}\|_{L^2(Y,\nu)} \left(\int_Y |k(x,y)|^2 d\nu(y) \right)^{1/2} \leq M \left(\int_Y |k(x,y)|^2 d\nu(y) \right)^{1/2} = H(x).$$

Note that by assumption, $H \in L^2(X,\mu)$. In other words, Kf_{n_k} converges pointwise μ-almost everywhere to $\int_Y k(x,y) f(y) d\nu(y)$, and it is bounded uniformly in k by the function $H \in L^2(X,\mu)$. We conclude by dominated convergence that

$$\lim_{k\to\infty} \|Kf_{n_k} - Kf\|^2_{L^2(X,\mu)} = \lim_{k\to\infty} \int_X \left|(Kf_{n_k})(x) - (Kf)(x)\right|^2 d\mu(x) = 0.$$