Problem 1. (i) By definition $O \cap X \subset X$ so that $\mathcal{T}_X \subset \mathcal{P}(X)$. $\emptyset \in \mathcal{T}$ implies $\emptyset \in \mathcal{T}_X$, and $S \in \mathcal{T}$ implies $X = S \cap X \in \mathcal{T}_X$. The finite intersection property holds in $\mathcal{T}_X$ since $\cap_{j=1}^n (O_j \cap X) = \cap_{j=1}^n O_j \cap X$ and the finite intersection property in $\mathcal{T}$. The arbitrary union property follows similarly from $\cup_{\alpha \in I} (O_\alpha \cap X) = (\cup_{\alpha \in I} O_\alpha) \cup X$.

(ii) Since $A$ is a subset of $X$, $A = A \cap X$ so that $A \in \mathcal{T} \Rightarrow A \in \mathcal{T}_X$. Reciprocally, if $A \in \mathcal{T}_X$, there exists $O \in \mathcal{T}$ such that $A = O \cap X$, showing that $A \in \mathcal{T}$ since both $O, X \in \mathcal{T}$.

(iii) Here, $A \subset X$ implies that $X \setminus A = X \cap (S \setminus A)$. Hence if $A$ is closed in $S$, then $X \setminus A \in \mathcal{T}_X$. Reciprocally, if $A$ is closed in $X$, there exists $O \in \mathcal{T}$ such that $X \setminus A = X \cap O = X \setminus (X \setminus O)$, namely $A = X \setminus O$, and further $A = (S \setminus O) \cap X$. Therefore, $A$ is the intersection of two closed sets in $S$, hence it is closed itself (indeed, if $C_1, C_2$ are closed then $C_1 \cap C_2 = (S \setminus O_1) \cap (S \setminus O_2) = S \setminus (O_1 \cup O_2)$ is closed).

Problem 2. (i) Clearly, $M \in \mathcal{T}$. Let $x \in \cap_{j=1}^n O_j$, with $O_j \in \mathcal{T}$. There exists $r_j$ such that $B_x(r_j) \subset O_j$ and hence $B_x(r_0) \subset \cap_{j=1}^n O_j$, where $r_0 = \min\{r_j : 1 \leq j \leq n\} > 0$ proving the finite intersection property. If now $x \in \cup_{\alpha \in I} O_\alpha$, there is $\alpha_0$ such that $x \in O_{\alpha_0}$ and hence $r_0$ such that $B_{r_0}(x) \subset O_{\alpha_0} \subset \cup_{\alpha \in I} O_\alpha$, proving the arbitrary union property.

(ii) Let $x \in M$. Then $\{B_q(x) : q \in \mathbb{Q}\}$ is a countable set. Let $N_x$ be a neighbourhood of $x$, namely $x \in N_x$. Since $N_x$ is open, $B_r(x) \subset N_x$ for some $r > 0$ and hence all $q \in \mathbb{Q}$ with $0 < q < r$. Hence, $\{B_q(x) : q \in \mathbb{Q}\}$ is a countable neighbourhood base for $x$, and $M$ is first countable since $x$ is arbitrary.

(iii) It suffices to show that separable implies second countable. Let $D \subset M$ be a countable dense set. Then $B = \{B_q(x) : q \in \mathbb{Q}, x \in D\}$ is countable base, so that $M$ is second countable. Indeed, let $O$ be open and $y \in O$. We show that there is $B \in \mathcal{B}$ such that $y \in B$. Clearly, $B_r(y) \subset O$ for some $r > 0$. For any $\epsilon > 0$ there is $x \in D$ such that $y \in B_x(x)$ by density. It follows that there is $0 < \delta \in \mathbb{Q}$ such that $y \in B_{\delta}(x) \subset B_r(y) \subset O$.

(iv) Let $x, y \in M$ be distinct. Then $3r := d(x, y) > 0$. Then $B_r(x) \cap B_r(y) = \emptyset$, proving the claim.

Problem 3. (i) $\emptyset \in \mathcal{T}$ and $S \setminus S = \emptyset$ is finite, hence $S \in \mathcal{T}$. If $Y_j \in \mathcal{T}$ for $1 \leq j \leq n$, then $S \setminus (\cap_{j=1}^n Y_j) = \cup_{j=1}^n (S \setminus Y_j)$ is finite, being a finite union of finite sets, hence $\cap_{j=1}^n Y_j \in \mathcal{T}$.

(ii) By definition, $S \setminus X$ and $S \setminus Y$ are finite, and hence so is their union $S \setminus (X \cap Y)$. Since $S$ is infinite, this implies that $X \cap Y$ is not empty (in fact, it is infinite).

(iii) Let $N_x = \{N_j : j \in \mathbb{N}\}$ be a countable base at $x$. Let $y \neq x$. Then $S \setminus \{y\}$ is an open neighbourhood of $y$ and hence $x \in N_{y_0} \subset S \setminus \{y\}$ for some $j_0$. Hence $y \notin \cap_{j=1}^\infty N_j$, and hence $\cap_{j=1}^\infty N_j = \{x\}$. But $S \setminus \{x\} = \cup_{j=1}^\infty (S \setminus N_j)$ is countable, since it is a countable union of finite sets. This is in contradiction of the uncountability of $S$.

(iv) Let $x \in S$ be arbitrary and let $O \in \mathcal{T}$ be such that $x \in O$. Then $S \setminus O$ is finite, and since $(x_n)_{n \in \mathbb{N}}$ does not take the same value twice, we conclude that $x_n \in O$ for all $n \geq n_x$ for some $n_x$. Hence $x_n \to x$ as $n \to \infty$.

Problem 4. (i) Let $x$ be a cluster point, and let $N_x$ be a countable neighbourhood base of $x$, such that $N_j \subset N_{j-1}$. For each $j$, let $x_{n_j} \in N_j$. Then $(x_{n_j})_{j \in \mathbb{N}}$ converges to $x$. Indeed, let $M_x$ be a neighbourhood of $x$ and let $N_k \subset M_x$. Then $x_{n_j} \in N_j \subset N_k$ for all $j \geq k$. Reciprocally, if $(x_{n_j})_{j \in \mathbb{N}}$ converges to $x$, then for any neighbourhood $N_x$ of $x$, $x_{n_j} \in N_x$ for all $j \geq j_0$. Hence $x$ is a cluster point since $\{j \geq j_0\}$ is infinite.

(ii) (a) Since $S \setminus \{0, 1\} = \{1\}$ is finite, $[0, 1)$ is open. Hence $S = [0, 1) \cup \{1\}$ is smallest closed set containing $[0, 1)$. (b) The set $V := S \setminus \{x_n : n \in \mathbb{N}\}$ of values of the sequence is open and $1 \in V$, so that it is a neighbourhood of 1. But $x_n \notin V$ for all $n \in \mathbb{N}$, showing that $(x_n)_{n \in \mathbb{N}}$ does not converge to 1. In particular, $((0, 1), T)$ is not first countable.
Problem 5. (i) \( S \) is disconnected if and only if there are disjoint, open \( U, V \neq \emptyset \) such that \( S \setminus V = U \), which is equivalent to \( U \) being both open and closed and \( U \neq S \) since \( V \neq \emptyset \).

(ii) Let \( E = \bigcup_{\alpha \in I} E_\alpha \). If \( E \) is disconnected, there are disjoint, nonempty, open (in \( E \)) sets \( U, V \) such that \( E = U \cup V \). By assumption, there exists \( x \in \bigcap_{\alpha \in I} E_\alpha \), w.l.o.g \( x \in U \). For \( y \in V \), we have \( y \in E_{\alpha_0} \) for some \( \alpha_0 \in I \). Of course, \( x \in E_{\alpha_0} \). Hence, \( E_{\alpha_0} \cap U \neq \emptyset \) as well as \( E_{\alpha_0} \cap V \neq \emptyset \), which is in contradiction with the fact that \( E_{\alpha_0} \) is connected.

(iii) Assume that \( \overline{X} \) is disconnected, namely \( \overline{X} = U \cup V \), with \( U, V \) nonempty open and closed in \( X \). If \( X \) is connected, then either \( X \cap U = \emptyset \) or \( X \cap V = \emptyset \), say the second one, namely \( X \subset U \). Taking the closure in \( X \) yields \( \overline{X} \subset U \), namely \( V = \emptyset \), which is a contradiction.

(iv) We declare \( x \sim y \) if there is a connected set containing both \( x, y \). Then \( \sim \) is an equivalence relation: indeed, if \( x \sim y \) and \( y \sim z \), then the union of the corresponding connected sets is connected by (ii). Let \( C_x \) be the equivalence class of \( x \), which is connected and maximal by construction. But \( C_x \subset \overline{C_x} \) which is connected by (iii), and hence by maximality \( C_x = \overline{C_x} \), proving that it is closed.