What is functional analysis?

- Study of topological spaces and of functional relations between them
- Study of spaces of functions
- Language of PDE, calculus of variations, integral equations
- Language of quantum mechanics

Functional analysis arose in the 19th century in a paradigmatic shift from the study of (the properties of) a single function/solution to the study of (the properties of) sets of functions/solutions and the relations between them. It is the language of much of modern mathematics, encompassing (linear) algebra, analysis and stochastic analysis.

Topics of the course:

- Topological spaces
- Normed linear spaces; as a running example: $L^p$-spaces
- Hilbert spaces
- Riesz’ representation theorem; as an application: Brownian motion
1. **Topological spaces**

Understanding limits and convergence is central to functional analysis. This ultimately has to do with the notions of open sets and neighbourhoods of a point. If the set is equipped with a distance, this can be done with open balls. In the more general setting of topological spaces, these concepts are introduced by the notion of a **topology**.

**Definition 1.1.** A **topological space** \((S, \mathcal{T})\) is a nonempty set \(S\) with a family of subsets \(\mathcal{T}\) such that

- \(\emptyset \in \mathcal{T}, \ S \in \mathcal{T}\)
- \(\mathcal{T}\) is closed under finite intersections:
  \[ A_1, \ldots A_n \in \mathcal{T} \Rightarrow \bigcap_{j=1}^{n} A_j \in \mathcal{T} \]
- \(\mathcal{T}\) is closed under arbitrary unions:
  \[ \{A_\alpha : \alpha \in I\} \subset \mathcal{T} \Rightarrow \bigcup_{\alpha \in I} A_\alpha \in \mathcal{T} \]

where \(I\) is an arbitrary index set.

The elements of \(\mathcal{T}\) are called the **open sets** of \(S\).

**Example 1.** (i) The discrete topology: \(\mathcal{T} = \mathcal{P}(S)\) the power set of \(S\), containing all subsets of \(S\)

(ii) The indiscrete topology: \(\mathcal{T} = \{\emptyset, S\}\)

(iii) Let \(S = \mathbb{R}^n\) with the elementary notion of open sets, namely \(X \in \mathcal{T}\) iff \(\forall x \in X, \exists r > 0\) s.t. \(\{y \in S : d(y, x) < r\} \subset X\), where \(d(\cdot,\cdot)\) is the Euclidean distance.

A **metric space** is a set \(M\) equipped with a function \(d : M \times M \to [0, \infty)\) such that

(i) \(d(x, y) = 0\) iff \(x = y\), (ii) \(d(x, y) = d(y, x)\), and (iii) \(d(x, z) \leq d(x, y) + d(y, z)\),
the triangle inequality. The metric defines a topology as in the third example above. Since any metric on $S$ gives rise to a topology, one may wonder whether every topology arises from a metric and the answer is, not surprisingly, no. If it is the case, $\mathcal{T}$ is called *metrizable*.

Topologies on a space $S$ can be ordered in a set-theoretic fashion: $\mathcal{T}_1 \prec \mathcal{T}_2$ iff $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_1$ is called *weaker* than $\mathcal{T}_2$.

Given a family $\mathcal{E} \subset \mathcal{P}(S)$, the unique weakest topology $\mathcal{T}(\mathcal{E})$ on $S$ containing $\mathcal{E}$ is called the topology *generated by* $\mathcal{E}$. It can be shown that $\mathcal{T}(\mathcal{E})$ consists of $\emptyset, S$ and all unions and all finite intersections of elements of $\mathcal{E}$.

**Definition 1.2.** A *base* of $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty $O \in \mathcal{T}$, there is a family $\{B_\alpha : \alpha \in I\} \subset \mathcal{B}$ and $O = \cup_{\alpha \in I} B_\alpha$.

If $(S, \mathcal{T})$ is a topological space, and $X \subset S$, then $\mathcal{T}_X := \{O \cap X : O \in \mathcal{T}\}$ defines a topology on $X$ called the *relative topology*.

The following concepts, familiar in $\mathbb{R}^n$, extend to general topological spaces. Let $X \subset S$.

- **$X$ is closed** if there is $Y \in \mathcal{T}$ such that $X = Y^c$
- The *interior* $X^\circ$ of $X$ is the largest open set contained in $X$
- The *closure* $\overline{X}$ of $X$ is the smallest closed set containing $X$
- The *boundary* $\partial X$ of $X$ is $\partial X = \overline{X} \setminus X^\circ$
- $X$ is called *dense in* $S$ if $\overline{X} = S$

A *neighbourhood* of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^\circ$. Note that a neighbourhood is not required to be open. A family $\mathcal{N}_x$ of subsets of $S$ is a *neighbourhood base at* $x$ if each $N \in \mathcal{N}_x$ is a neighbourhood of $x$ and if for any neighbourhood $M_x$ of $x$, there is an $N \in \mathcal{N}_x$ such that $N \subset M_x$. 


There are two major classifications of topological spaces. The first one is about how well open sets separate points. While the classification has five classes denoted $T_0, \ldots, T_4$, we only introduce the following, which plays an important role in the discussion of compactness.

**Definition 1.3.** A topological space $(S, \mathcal{T})$ is called Hausdorff, or $T_2$, if for all pairs $x, y \in S, x \neq y$, $\exists O_x, O_y \in \mathcal{T}$ with $O_x \cap O_y = \emptyset$, such that $x \in O_x, y \in O_y$.

The second classification is about countability and it is particularly relevant in discussing questions of convergence (and consequently its relation to compactness).

**Definition 1.4.** A topological space $(S, \mathcal{T})$ is called

- separable if it has a countable dense set
- first countable if each $x \in S$ has a countable neighbourhood base
- second countable if $S$ has a countable base

**Proposition 1.5.** (i) Second countable $\Rightarrow$ First countable

(ii) Second countable $\Rightarrow$ Separable

**Proof.** Let $\mathcal{B}$ be a countable base of $\mathcal{T}$.

(i) For any $x \in S$, the family $\mathcal{N}_x := \{ N \in \mathcal{B} : x \in N \}$ is a countable neighbourhood base at $x$. Indeed, if $M_x$ is a neighbourhood of $x$, then $\bigcup_j N_j = M_x^c \subset M_x$, where $N_j \in \mathcal{B}$. Hence there is $j_0$ such that $x \in N_{j_0} \subset M_x$, and $N_{j_0} \in \mathcal{N}_x$.

(ii) For each $B \in \mathcal{B}$, let $x_B \in B$. Then the set $D := \{ x_B : B \in \mathcal{B} \}$ is countable. But $\overline{D}^c$ is open by construction it does not include any $B \in \mathcal{B}$. It follows from the definition of a base that $\overline{D}^c = \emptyset$, namely, $D$ is dense. $\square$
Note that there are separable spaces that are not second countable.

**Example 2.** Consider $\mathbb{R}^n$ equipped with the usual topology. Then the family of all open balls (any centre, any radius) is a base. For any $x \in \mathbb{R}^n$ the family $\{B_{p/q}(x) : p, q \in \mathbb{N}\}$ of closed balls for rational radii is a neighbourhood base. Hence $\mathbb{R}^n$ is first countable.

This again generalizes to general metric spaces. A metric space is first countable. Moreover, a metric space is second countable iff it is separable.

We are now ready to turn to the general notion of convergence.

**Definition 1.6.** A sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space $(S, T)$ is **convergent** if there is $x \in S$ such that for every neighbourhood $N_x$ of $x$, there is $n_0$ such that $x_n \in N_x$ for all $n \geq n_0$.

Here is a first result that is valid only in first countable spaces, namely that the closure of a subset is given by the set of limit points of sequences.

**Proposition 1.7.** Let $(S, T)$ be a first countable topological space and $X \subset S$. Then $x \in \overline{X}$ iff $x$ is the limit of a convergent sequence $(x_n)_{n \in \mathbb{N}}$ in $X$.

**Proof.** Let $\mathcal{N}_x := \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of $x$ such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then for any $n \in \mathbb{N}$, $O_n \cap X \neq \emptyset$ (since otherwise $x \notin (O_n^c)^c$ would be a closed set containing $X$, but $x \in \overline{X} \subset (O_n^c)^c$ is a contradiction) and we can pick $x_n \in O_n \cap X$. This is a convergent sequence such that $\lim_{n \to \infty} x_n = x$. Reciprocally, assume that $x \in (X)^c$. For any sequence $(y_n)_{n \in \mathbb{N}}$ in $X$, the open neighbourhood $(X)^c$ contains no point of the sequence, and hence $(y_n)_{n \in \mathbb{N}}$ does not converge to $x$. $\Box$
Note that if $M_x := \{U_n : n \in \mathbb{N}\}$ is any a countable neighbourhood base at $x$, the sets $O_j = \cap_{n=1}^j U_j$ form a ‘decreasing’ neighbourhood base as used in the proof.

If $(S, \mathcal{T})$ is not first countable, this criterion is not sufficient. The closure is given by limit points of nets, which are generalizations of sequences of the form $(x_\alpha)_{\alpha \in I}$ where $I$ is not necessarily countable and only partially ordered.

We are equipped to turn to continuity.

**Definition 1.8.** Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is **continuous** if $f^{-1}(O) \in \mathcal{T}_1$ for any $O \in \mathcal{T}_2$.

In other words, the preimage of any open set is open. This should not be confused with the following:

**Definition 1.9.** Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is **open** if $f(O) \in \mathcal{T}_2$ for any $O \in \mathcal{T}_1$.

An invertible function that is both open and continuous is a **homeomorphism**.

While continuity is defined in terms of two topologies, one can reciprocally use continuity to define topologies. Let $S_1$ be a set (not yet equipped with a topology) and let $(S_2, \mathcal{T}_2)$ be a topological space. Let $\mathcal{F}$ be a family of functions from $S_1$ to $S_2$. Then the topology on $S_1$ generated by $\{f^{-1}(O) : O \in \mathcal{T}_2\}$ is called the $\mathcal{F}$-weak topology. By definition, all functions $f \in \mathcal{F}$ are continuous with respect to this topology on $S_1$.

**Example 3.** Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \to S_2, E_x(f) = f(x)$ be the evaluation functions and let $\mathcal{F} = \{E_x : x \in [a, b]\}$. The $\mathcal{F}$-weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.
Let us turn to compactness. In a topological space \((S, \mathcal{T})\), an open cover is a family \(\mathcal{C} \subset \mathcal{T}\) such that \(S = \bigcup_{O \in \mathcal{C}} O\). A subcover is a subset of \(\mathcal{C}\) that is a cover.

**Definition 1.10.** A topological space \((S, \mathcal{T})\) is compact if any open cover has a finite subcover.

A subset \(X \subset S\) is a compact set if it is compact in the relative topology. It is called precompact if its closure is compact.

Compactness can also be formulated in terms of closed sets. \((S, \mathcal{T})\) is said to have the finite intersection property if any family \(\mathcal{C}\) of closed set such that \(\bigcap_{j=1}^{n} C_j \neq \emptyset\) for any finite subfamily \(\{C_1, \ldots, C_n\} \subset \mathcal{F}\) satisfies \(\bigcap_{C \in \mathcal{C}} C \neq \emptyset\). We then have the following result: \(S\) is compact if and only if \(S\) has the finite intersection property.

**Proposition 1.11.** Let \(X \subset S\) be a subset of a compact topological space \((S, \mathcal{T})\). If \(X\) is closed, then it is compact (in the relative topology).

**Proof.** Let \(\mathcal{C}\) be an open cover of \(X\). By the definition of the relative topology, any \(C \in \mathcal{C}\) is of the form \(O_C \cap X\) with \(O_C \in \mathcal{T}\). If \(\mathcal{O}\) is the set of these \(O_C\)'s, then \(\mathcal{O} \cup \{X^c\}\) is an open cover of \(S\) since \(X\) is closed. \(S\) being compact, there is a finite subcover \(\hat{\mathcal{O}}\), which yields, by intersecting with \(X\), a finite open cover \(\hat{\mathcal{C}}\) of \(X\). \(\square\)

Another useful result is that compactness is pushed forward by continuous functions. It in particular generalizes the well-known fact that a continuous, real-valued function defined on a compact interval reaches its maximum and minimum values.

**Proposition 1.12.** Let \((S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)\) be topological spaces, and let \(f : S_1 \to S_2\) be a continuous function. If \(S_1\) is compact, then \(f(S_1) \subset S_2\) is compact.
Proof. Let $C = \{C_{\alpha} : \alpha \in I\}$ be an open cover of $f(S_1) \subset S_2$ in the relative topology. There are open sets $\{O_{\alpha} : \alpha \in I\}$ in $S_2$ such that $C_{\alpha} = O_{\alpha} \cap f(S_1)$, and $f^{-1}(O_{\alpha})$ is open in $S_1$ by continuity. Therefore, $\{f^{-1}(O_{\alpha}) : \alpha \in I\}$ is an open cover of $S_1$, from which one can extract a finite subcover $\{f^{-1}(O_n) : 1 \leq n \leq N\}$. But then $C_n = O_n \cap f(S_1) : 1 \leq n \leq N$ is a finite subcover of $C$. □

It is worth pointing out that the Bolzano-Weierstrass theorem of real analysis does not hold in a general topological space. In fact, one must consider nets instead of sequences. However it does in a second countable space:

**Theorem 1.13.** A second countable topological space $(S, T)$ is compact iff every sequence has a convergent subsequence.

**Proof.** Assume that $S$ is compact, let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $S$ that does not have a convergent subsequence. Since $S$ is second countable, it is first countable, so that $(z_n)_{n \in \mathbb{N}}$ does not have a cluster point (see Problem 4(i), Sheet 1). Hence, for any $x \in S$, there is an open set $O_x \ni x$ such that $z_n \in O_x$ for only finitely many $n$’s. In particular, there is $n_x \in \mathbb{N}$ such that $z_n \not\in O_x$ for all $n \geq n_x$. Extracting a finite cover $\{O_{x_i} : 1 \leq i \leq N\}$ from $\{O_x : x \in S\}$, and letting $n_0 = \max \{n_{x_i} : 1 \leq i \leq N\}$, we have that $z_n \not\in \bigcup_{i=1}^{N} O_{x_i} = S$ for all $n \geq n_0$, a contradiction.

Reciprocally, assume that every sequence has a convergent subsequence. Since $S$ is second countable, it has a countable open cover $C = \{O_j : j \in \mathbb{N}\}$. Assume that there is no finite subcover of $C$. Then for any $n \in \mathbb{N}$, there is $x_n \not\in \bigcup_{j=1}^{n} O_j$. Let $(x_{n_k})_{k \in \mathbb{N}}$ be a convergent subsequence and let $x$ be its limit. Since $C$ is a cover, there is $j_0$ such that $x \in O_{j_0}$, and hence there is $k_0$ such that $x_{n_k} \in O_{j_0}$ for all $k \geq k_0$. This is contradiction with $x_{n_k} \not\in \bigcup_{j=1}^{n_k} O_j$ for any $n_k > j_0$. □
The property that every sequence has a convergent subsequence is called *sequential compactness*. The first part of the theorem shows that compactness implies sequential compactness in a first countable space (a fortiori in a second countable space and in a metric space).

We now turn to the the Stone-Weierstrass theorem. First of all, we recall the ‘classical’ Weierstrass theorem:

**Proposition 1.14.** If $f$ is a continuous complex-valued function on $[a, b]$, then there exists a sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} P_n = f$$

uniformly on $[a, b]$. If $f$ is real-valued, then the $P_n$’s may be taken real.

In other words, the polynomials are dense in the set $C([a, b])$ of continuous functions on the compact interval $[a, b]$. The Stone-Weierstrass theorem generalizes the result to an arbitrary compact Hausdorff space.

Let $X$ be a compact Hausdorff space. We first note that $C_\mathbb{R}(X)$, the real-valued continuous functions on $X$ equipped with the multiplication $(fg)(x) = f(x)g(x)$ is an algebra.

We say that a subalgebra $\mathcal{A}$ of $C_\mathbb{R}(X)$ *separates points* if $x, y \in X$ such that $x \neq y$ implies $\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

**Theorem 1.15.** Let $X$ be a compact Hausdorff space. Let $\mathcal{A}$ be a closed (with respect to $\| \cdot \|_\infty$) subalgebra of $C_\mathbb{R}(X)$ that separates points. Then either $\mathcal{A} = C_\mathbb{R}(X)$ or $\exists x_0 \in X$ such that $\mathcal{A} = \{ f \in C_\mathbb{R}(X) : f(x_0) = 0 \}$.