What is functional analysis?

- Study of topological spaces and of functional relations between them
- Study of spaces of functions
- Language of PDE, calculus of variations, integral equations
- Language of quantum mechanics

Functional analysis arose in the 19th century in a paradigmatic shift from the study of (the properties of) a single function/solution to the study of (the properties of) sets of functions/solutions and the relations between them. It is the language of much of modern mathematics, encompassing (linear) algebra, analysis and stochastic analysis.

Topics of the course:

- Topological spaces
- Normed linear spaces; as a running example: $L^p$-spaces
- Hilbert spaces
- Riesz’ representation theorem; as an application: Brownian motion
1. **Topological spaces**

Understanding limits and convergence is central to functional analysis. This ultimately has to do with the notions of open sets and neighbourhoods of a point. If the set is equipped with a distance, this can be done with open balls. In the more general setting of topological spaces, these concepts are introduced by the notion of a *topology*.

**Definition 1.1.** A topological space \((S, \mathcal{T})\) is a nonempty set \(S\) with a family of subsets \(\mathcal{T}\) such that

1. \(\emptyset \in \mathcal{T}, S \in \mathcal{T}\)
2. \(\mathcal{T}\) is closed under finite intersections:
   \[A_1, \ldots, A_n \in \mathcal{T} \Rightarrow \bigcap_{j=1}^{n} A_j \in \mathcal{T}\]
3. \(\mathcal{T}\) is closed under arbitrary unions:
   \[\{A_\alpha : \alpha \in I\} \subset \mathcal{T} \Rightarrow \bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}\]

   where \(I\) is an arbitrary index set.

The elements of \(\mathcal{T}\) are called the *open sets* of \(S\).

**Example 1.** (i) The discrete topology: \(\mathcal{T} = \mathcal{P}(S)\) the power set of \(S\), containing all subsets of \(S\)

(ii) The indiscrete topology: \(\mathcal{T} = \{\emptyset, S\}\)

(iii) Let \(S = \mathbb{R}^n\) with the elementary notion of open sets, namely \(X \in \mathcal{T}\) iff \(\forall x \in X, \exists r > 0\) s.t. \(\{y \in S : d(y, x) < r\} \subset X\), where \(d(\cdot, \cdot)\) is the Euclidean distance.

A *metric space* is a set \(M\) equipped with a function \(d : M \times M \to [0, \infty)\) such that

(i) \(d(x, y) = 0\) iff \(x = y\),
(ii) \(d(x, y) = d(y, x)\), and
(iii) \(d(x, z) \leq d(x, y) + d(y, z)\),
the triangle inequality. The metric defines a topology as in the third example above. Since any metric on $S$ gives rise to a topology, one may wonder whether every topology arises from a metric and the answer is, not surprisingly, no. If it is the case, $T$ is called **metrizable**.

Topologies on a space $S$ can be ordered in a set-theoretic fashion: $T_1 \prec T_2$ iff $T_1 \subset T_2$ and $T_1$ is called *weaker* than $T_2$.

Given a family $E \subset \mathcal{P}(S)$, the unique weakest topology $T(E)$ on $S$ containing $E$ is called the topology *generated by* $E$. It can be shown that $T(E)$ consists of $\emptyset, S$ and all unions and all finite intersections of elements of $E$.

**Definition 1.2.** A base of $T$ is a family $B \subset T$ such that for any nonempty $O \in T$, there is a family $\{B_\alpha : \alpha \in I\} \subset B$ and $O = \cup_{\alpha \in I} B_\alpha$.

If $(S, T)$ is a topological space, and $X \subset S$, then $T_X := \{O \cap X : O \in T\}$ defines a topology on $X$ called the relative topology.

The following concepts, familiar in $\mathbb{R}^n$, extend to general topological spaces. Let $X \subset S$.

- $X$ is closed if there is $Y \in T$ such that $X = Y^c$
- The interior $X^o$ of $X$ is the largest open set contained in $X$
- The closure $\overline{X}$ of $X$ is the smallest closed set containing $X$
- The boundary $\partial X$ of $X$ is $\partial X = \overline{X} \setminus X^o$
- $X$ is called dense in $S$ if $\overline{X} = S$

A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^o$. Note that a neighbourhood is not required to be open. A family $\mathcal{N}_x$ of subsets of $S$ is a neighbourhood base at $x$ if each $N \in \mathcal{N}_x$ is a neighbourhood of $x$ and if for any neighbourhood $M_x$ of $x$, there is an $N \in \mathcal{N}_x$ such that $N \subset M_x$. 

There are two major classifications of topological spaces. The first one is about how well open sets separate points. While the classification has five classes denoted $T_0, \ldots, T_4$, we only introduce the following, which plays an important role in the discussion of compactness.

**Definition 1.3.** A topological space $(S, \mathcal{T})$ is called Hausdorff, or $T_2$, if for all pairs $x, y \in S, x \neq y$, $\exists O_x, O_y \in \mathcal{T}$ with $O_x \cap O_y = \emptyset$, such that $x \in O_x, y \in O_y$.

The second classification is about countability and it is particularly relevant in discussing questions of convergence (and consequently its relation to compactness).

**Definition 1.4.** A topological space $(S, \mathcal{T})$ is called

- **separable** if it has a countable dense set
- **first countable** if each $x \in S$ has a countable neighbourhood base
- **second countable** if $S$ has a countable base

**Proposition 1.5.** (i) Second countable $\Rightarrow$ First countable

(ii) Second countable $\Rightarrow$ Separable

*Proof.* Let $\mathcal{B}$ be a countable base of $\mathcal{T}$.

(i) For any $x \in S$, the family $\mathcal{N}_x := \{ N \in \mathcal{B} : x \in N \}$ is a countable neighbourhood base at $x$. Indeed, if $M_x$ is a neighbourhood of $x$, then $\bigcup_j N_j = M_x^c \subset M_x$, where $N_j \in \mathcal{B}$. Hence there is $j_0$ such that $x \in N_{j_0} \subset M_x$, and $N_{j_0} \in \mathcal{N}_x$.

(ii) For each $B \in \mathcal{B}$, let $x_B \in B$. Then the set $D := \{x_B : B \in \mathcal{B}\}$ is countable. But $\overline{D}^c$ is open by construction it does not include any $B \in \mathcal{B}$. It follows from the definition of a base that $\overline{D}^c = \emptyset$, namely, $D$ is dense. $\square$
Note that there are separable spaces that are not second countable.

**Example 2.** Consider $\mathbb{R}^n$ equipped with the usual topology. Then the family of all open balls (any centre, any radius) is a base. For any $x \in \mathbb{R}^n$ the family $\{B_{p/q}(x) : p, q \in \mathbb{N}\}$ of closed balls for rational radii is a neighbourhood base. Hence $\mathbb{R}^n$ is first countable.

This again generalizes to general metric spaces. A metric space is first countable. Moreover, a metric space is second countable iff it is separable.

We are now ready to turn to the general notion of convergence.

**Definition 1.6.** A sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space $(S, T)$ is convergent if there is $x \in S$ such that for every neighbourhood $N_x$ of $x$, there is $n_0$ such that $x_n \in N_x$ for all $n \geq n_0$.

Here is a first result that is valid only in first countable spaces, namely that the closure of a subset is given by the set of limit points of sequences.

**Proposition 1.7.** Let $(S, T)$ be a first countable topological space and $X \subset S$. Then $x \in \overline{X}$ iff $x$ is the limit of a convergent sequence $(x_n)_{n \in \mathbb{N}}$ in $X$.

**Proof.** Let $\mathcal{N}_x := \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of $x$ such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then for any $n \in \mathbb{N}$, $O_n \cap X \neq \emptyset$ (since otherwise $x \notin (O_n^c)^c$ would be a closed set containing $X$, but $x \in \overline{X} \subset (O_n^c)^c$ is a contradiction) and we can pick $x_n \in O_n \cap X$. This is a convergent sequence such that $\lim_{n \to \infty} x_n = x$. Reciprocally, assume that $x \in (\overline{X})^c$. For any sequence $(y_n)_{n \in \mathbb{N}}$ in $X$, the open neighbourhood $(\overline{X})^c$ contains no point of the sequence, and hence $(y_n)_{n \in \mathbb{N}}$ does not converge to $x$. □
Note that if $M_x := \{U_n : n \in \mathbb{N}\}$ is any a countable neighbourhood base at $x$, the sets $O_j = \bigcap_{n=1}^j U_j$ form a ‘decreasing’ neighbourhood base as used in the proof.

If $(S, T)$ is not first countable, this criterion is not sufficient. The closure is given by limit points of nets, which are generalizations of sequences of the form $(x_\alpha)_{\alpha \in I}$ where $I$ is not necessarily countable and only partially ordered.

We are equipped to turn to continuity.

**Definition 1.8.** Let $(S_1, T_1), (S_2, T_2)$ be topological spaces. A function $f : S_1 \to S_2$ is *continuous* if $f^{-1}(O) \in T_1$ for any $O \in T_2$.

In other words, the preimage of any open set is open. This should not be confused with the following:

**Definition 1.9.** Let $(S_1, T_1), (S_2, T_2)$ be topological spaces. A function $f : S_1 \to S_2$ is *open* if $f(O) \in T_2$ for any $O \in T_1$.

An invertible function that is both open and continuous is a *homeomorphism*.

While continuity is defined in terms of two topologies, one can reciprocally use continuity to define topologies. Let $S_1$ be a set (not yet equipped with a topology) and let $(S_2, T_2)$ be a topological space. Let $F$ be a family of functions from $S_1$ to $S_2$. Then the topology on $S_1$ generated by $\{f^{-1}(O) : O \in T_2\}$ is called the $F$-weak topology. By definition, all functions $f \in F$ are continuous with respect to this topology on $S_1$.

**Example 3.** Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \to S_2, E_x(f) = f(x)$ be the evaluation functions and let $F = \{E_x : x \in [a, b]\}$. The $F$-weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.
Let us turn to compactness. In a topological space $(S, \mathcal{T})$, an open cover is a family $\mathcal{C} \subset \mathcal{T}$ such that $S = \bigcup_{O \in \mathcal{C}} O$. A subcover is a subset of $\mathcal{C}$ that is a cover.

**Definition 1.10.** A topological space $(S, \mathcal{T})$ is compact if any open cover has a finite subcover.

A subset $X \subset S$ is a compact set if it is compact in the relative topology. It is called precompact if its closure is compact. Note that if a family of open sets $\mathcal{C} = \{O_{\alpha} \in \mathcal{T} : \alpha \in I\}$ is such that $X \subset \bigcup_{\alpha \in I} O_{\alpha}$, then $\mathcal{C}_X = \{O_{\alpha} \cap X \in \mathcal{T} : \alpha \in I\}$ is an open cover of $X$. This is usually how open covers of subsets are constructed.

Compactness can also be formulated in terms of closed sets. $(S, \mathcal{T})$ is said to have the finite intersection property if any family $\mathcal{C}$ of closed sets such that $\bigcap_{j=1}^{n} C_{j} \neq \emptyset$ for any finite subfamily $\{C_{1}, \ldots, C_{n}\} \subset \mathcal{F}$ satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. We then have the following result:

$S$ is compact iff $S$ has the finite intersection property.

**Proposition 1.11.** Let $X \subset S$ be a subset of a compact topological space $(S, \mathcal{T})$. If $X$ is closed, then it is compact (in the relative topology).

**Proof.** Let $\mathcal{C}$ be an open cover of $X$. By the definition of the relative topology, any $C \in \mathcal{C}$ is of the form $O_{C} \cap X$ with $O_{C} \in \mathcal{T}$. If $\mathcal{O}$ is the set of these $O_{C}$’s, then $\mathcal{O} \cup \{X^{c}\}$ is an open cover of $S$ since $X$ is closed. $S$ being compact, there is a finite subcover $\tilde{\mathcal{O}}$, which yields, by intersecting with $X$, a finite open cover $\tilde{\mathcal{C}}$ of $X$. □

Another useful result is that compactness is pushed forward by continuous functions. It in particular generalizes the well-known fact that a continuous, real-valued function defined on a compact interval reaches it maximum and minimum values.

**Proposition 1.12.** Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces, and let $f : S_1 \to S_2$ be a continuous function. If $S_1$ is compact, then $f(S_1) \subset S_2$ is compact.
Proof. Let \( C = \{ C_\alpha : \alpha \in I \} \) be an open cover of \( f(S_1) \subset S_2 \) in the relative topology. There are open sets \( \{ O_\alpha : \alpha \in I \} \) in \( S_2 \) such that \( C_\alpha = O_\alpha \cap f(S_1) \), and \( f^{-1}(O_\alpha) \) is open in \( S_1 \) by continuity. Therefore, \( \{ f^{-1}(O_\alpha) : \alpha \in I \} \) is an open cover of \( S_1 \), from which one can extract a finite subcover \( \{ f^{-1}(O_n) : 1 \leq n \leq N \} \). But then \( \{ C_n = O_n \cap f(S_1) : 1 \leq n \leq N \} \) is a finite subcover of \( C \). □

It is worth pointing out that the Bolzano-Weierstrass theorem of real analysis does not hold in a general topological space. In fact, one must consider nets instead of sequences. However it does in a second countable space:

**Theorem 1.13.** A second countable topological space \((S, T)\) is compact iff every sequence has a convergent subsequence.

**Proof.** Assume that \( S \) is compact, let \((z_n)_{n \in \mathbb{N}}\) be a sequence in \( S \) that does not have a convergent subsequence. Since \( S \) is second countable, it is first countable, so that \((z_n)_{n \in \mathbb{N}}\) does not have a cluster point (see Problem 4(i), Sheet 1). Hence, for any \( x \in S \), there is an open set \( O_x \ni x \) such that \( z_n \in O_x \) for only finitely many \( n \)'s. In particular, there is \( n_x \in \mathbb{N} \) such that \( z_n \notin O_x \) for all \( n \geq n_x \). Extracting a finite cover \( \{ O_i : 1 \leq i \leq N \} \) from \( \{ O_x : x \in S \} \), and letting \( n_0 = \max\{ n_{x_i} : 1 \leq i \leq N \} \), we have that \( z_n \notin \bigcup_{i=1}^{N} O_{x_i} = S \) for all \( n \geq n_0 \), a contradiction.

Reciprocally, assume that every sequence has a convergent subsequence. Since \( S \) is second countable, it has a countable open cover \( C = \{ O_j : j \in \mathbb{N} \} \). Assume that there is no finite subcover of \( C \). Then for any \( n \in \mathbb{N} \), there is \( x_n \notin \bigcup_{j=1}^{n} O_j \). Let \((x_{nk})_{k \in \mathbb{N}}\) be a convergent subsequence and let \( x \) be its limit. Since \( C \) is a cover, there is \( j_0 \) such that \( x \in O_{j_0} \), and hence there is \( k_0 \) such that \( x_{nk} \in O_{j_0} \) for all \( k \geq k_0 \). This is contradiction with \( x_{nk} \notin \bigcup_{j=1}^{k_0} O_j \) for any \( n_k > j_0 \). □
The property that every sequence has a convergent subsequence is called sequential compactness. The first part of the theorem shows that compactness implies sequential compactness in a first countable space (a fortiori in a second countable space and in a metric space).

**Lemma 1.14.** Let \((S, \mathcal{T})\) be a Hausdorff space. Let \((x_n)_{n \in \mathbb{N}}\) be a convergent sequence in \(S\). Then the limit \(x = \lim_{n \to \infty} x_n\) is unique.

**Proof.** Let \(x = \lim_{n \to \infty} x_n\) and let \(y \neq x\). There exist disjoint \(O_x, O_y \in \mathcal{T}\) with \(x \in O_x, y \in O_y\). But \(x_n \to x\) implies that there is \(n_0\) such that \(x_n \in O_x\) for all \(n \geq n_0\), and in particular \(x_n \notin O_y, n \geq n_0\). It follows that \((x_n)_{n \in \mathbb{N}}\) does not converge to \(y\). \(\square\)

**Theorem 1.15.** Let \((S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)\) be two compact Hausdorff spaces and let \(f : S_1 \to S_2\) be a continuous bijection. The \(f\) is a homeomorphism.

The proof relies on the proposition of the following separation lemma.

**Proposition 1.16.** Let \((S, \mathcal{T})\) be a Hausdorff space and let \(K\) be a compact subset of \(S\). Then \(K\) is closed.

**Proof.** For any \(x \in X^c\), there is an open \(U_x \ni x\) such that \(K \cap U = \emptyset\), see the lemma below. Hence \(X^c = \bigcup_{x \in X} U_x\) is open. \(\square\)

**Lemma 1.17.** Let \((S, \mathcal{T})\) be a Hausdorff space and let \(K\) be a compact subset of \(S\). For any \(x \in K^c\), there are disjoint open sets \(U, V\) such that \(x \in U, K \subset V\).

**Proof.** Let \(x \in K^c, y \in K\). There are disjoint open \(U_y, O_y\) such that \(x \in U_y, y \in O_y\). Using the open cover \(\{O_y : y \in K\}\), there are \(\{y_1, \ldots, y_N\}\) in \(K\) such that

\[K \subset \bigcup_{j=1}^N O_{y_j} = V.\]
Moreover, the set \( U = \cap_{j=1}^{N} U_{y_{j}} \) contains \( x \) and is disjoint from \( V \). \qed

We can now prove Theorem 1.15.

\textit{Proof.} We prove that \( f \) is open. It suffices to show that \( f(C) \in S_{2} \) is closed whenever \( C \subset S_{1} \) is closed. Since \( S_{1} \) is compact, \( C \) is compact by Proposition 1.11. Therefore, \( f(C) \) is compact by Proposition 1.12 and hence closed since \( S_{2} \) is Hausdorff. \qed

We now turn to the the Stone-Weierstrass theorem. First of all, we recall the ‘classical’ Weierstrass theorem:

\textbf{Proposition 1.18.} If \( f \) is a continuous real-valued function on \([a, b]\), then there exists a sequence of polynomials \((P_{n})_{n \in \mathbb{N}}\) such that

\[
\lim_{n \to \infty} P_{n} = f
\]

uniformly on \([a, b]\).

In other words, the polynomials are dense in the set \( C_{\mathbb{R}}([a, b]) \) of continuous real-valued functions on the compact interval \([a, b]\). The Stone-Weierstrass theorem generalizes the result to an arbitrary compact Hausdorff space.

Let \( X \) be a compact Hausdorff space. We first note that \( C_{\mathbb{R}}(X) \), the real-valued continuous functions on \( X \) equipped with the multiplication \((fg)(x) = f(x)g(x)\) is an algebra. We say that a subalgebra \( A \) of \( C_{\mathbb{R}}(X) \) \textit{separates points} if \( x, y \in X \) such that \( x \neq y \) implies \( \exists f \in A \) such that \( f(x) \neq f(y) \).

\textbf{Theorem 1.19.} Let \( X \) be a compact Hausdorff space. Let \( A \) be a closed (with respect to \( \| \cdot \|_{\infty} \)) subalgebra of \( C_{\mathbb{R}}(X) \) that separates points. Then either \( A = C_{\mathbb{R}}(X) \) or \( \exists x_{0} \in X \) such that \( A = \{ f \in C_{\mathbb{R}}(X) : f(x_{0}) = 0 \} \).
In particular, if $1 \in \mathcal{A}$, then the second case is excluded; there is no proper closed unital subalgebra of $C_\mathbb{R}(X)$ that separates points. We prove the theorem in this slightly easier case. Note that if $\mathcal{A}$ is not closed, the theorem applies to $\overline{\mathcal{A}}$ in which case it can be stated as: Any unital subalgebra $\mathcal{A}$ that separates points is dense in $C_\mathbb{R}(X)$ in the uniform topology.

We note that Hausdorffness is not used in the proof. However, it is a necessary condition for the existence of an algebra separating points. Indeed, if there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$, then $f(x), f(y)$ have disjoint open neighbourhoods (since $\mathbb{R}$ is Hausdorff) and their preimages must be disjoint open neighbourhoods of $x$, repsectively $y$.

The proof uses the concept of a lattice: A subset $\mathcal{F} \subset C_\mathbb{R}(X)$ is called a lattice if for all $f, g \in \mathcal{F}$, the functions $f \wedge g := \min\{f, g\}$ and $f \vee g := \max\{f, g\}$ are in $\mathcal{F}$.

Lemma 1.20. Any closed unital subalgebra $\mathcal{A}$ of $C_\mathbb{R}(X)$ is a lattice.

Proof. Since

$$f \vee g = \frac{1}{2}|f - g| + \frac{1}{2}(f + g), \quad f \wedge g = -((-f) \vee (-g)),$$

it suffices to prove that $f \in \mathcal{A}$ implies $|f| \in \mathcal{A}$. Since there is nothing to prove is $f = 0$, we assume that $f \neq 0$. Since $f$ is continuous on a compact $X$, it is bounded, namely $\|f\|_\infty = \sup_{x \in X} |f(x)| < \infty$. By the classical Weierstrass theorem, there is a sequence of polynomials such that $|P_n(x) - |x|| < n^{-1}$ for all $x \in [-1, 1]$. Hence

$$\|P_n(h) - |h|\|_\infty < \frac{1}{n},$$

where $h = f/\|f\|_\infty$, namely $P_n(h) \to |h|$ uniformly. Since $\mathcal{A}$ is a unital algebra, $f \in \mathcal{A}$ implies $P_n(h) \in \mathcal{A}$, and the convergence just proved concludes the proof since $\mathcal{A}$ is closed w.r.t. $\|\cdot\|_\infty$. \qed

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The final part of the proof goes by the name of Kakutani-Krein theorem.

**Proposition 1.21.** Let \( \mathcal{L} \subset C_\mathbb{R}(X) \) be a closed lattice that contains 1 and that separates points. Then \( \mathcal{L} = C_\mathbb{R}(X) \).

**Proof.** Let \( g \in C_\mathbb{R}(X) \). Let \( x \neq y \) and let \( \epsilon > 0 \). The map \( \mathcal{L} \ni h \mapsto (h(x), h(y)) \in \mathbb{R}^2 \) is an algebra homomorphism (\( \mathbb{R}^2 \) under coordinatewise addition and multiplication), the range of which contains \((1, 1)\) since \( 1 \in \mathcal{L} \) as well as one element of the form \((a, b)\) with \( a \neq b \) since \( \mathcal{L} \) separates points. Hence its range is all of \( \mathbb{R}^2 \), so that there is \( f_{xy} \in \mathcal{L} \) such that \( f_{xy}(x) = g(x), f_{xy}(y) = g(y) \).

(We first consider \( x \) fixed and \( y \) arbitrary) By continuity, there is a neighbourhood \( N_y \) of \( y \) such that \( f_{xy}(z) + \epsilon > g(z) \) for all \( z \in N_y \). By compactness, there is a finite set \( \{y_1, \ldots, y_n\} \) such that \( \{N_{y_j} : 1 \leq j \leq n\} \) is a subcover of \( X \). The function \( f_x := f_{xy_1} \lor \cdots \lor f_{xy_n} \), is such that \( f_x(x) = g(x) \) and \( f_x(z) + \epsilon > g(z) \) for all \( z \in X \).

(We now consider \( x \) arbitrary) Similarly, there is a neighbourhood \( M_x \ni x \) such that \( f_x(z) - \epsilon < g(z) \) for all \( z \in M_x \). Extracting a finite subcover indexed by \( \{x_1, \ldots, x_m\} \) and letting \( f := f_{x_1} \land \cdots \land f_{x_m} \), we conclude that \( f(z) - \epsilon < g(z) \) for all \( z \in X \). By the previous part \( f(z) + \epsilon > g(z) \), so that we have constructed \( f \in \mathcal{L} \) such that \( \|f - g\|_\infty < \epsilon \). Since \( \epsilon \) is arbitrary, this shows that \( \mathcal{L} \) is dense and hence equal to \( C_\mathbb{R}(X) \) because it is closed. \( \Box \)

The Stone-Weierstrass extends is two directions. First of all, it extend to complex-valued functions, provided the subalgebra \( \mathcal{A} \) is closed under complex conjugation, namely \( f \in \mathcal{A} \) implies \( \bar{f} \in \mathcal{A} \) (and indeed, the result is in general false). Indeed, any \( f \in C_\mathbb{C}(X) \) can be written as \( f = \frac{(f + \bar{f})}{2} - i \frac{(f - \bar{f})}{2} \), where both terms are in \( \mathcal{A} \cap C_\mathbb{R}(X) \). The complex
Stone-Weierstrass theorem follows from an application of the real one to the real and imaginary parts of \( f \).

Secondly, it extends to locally compact Hausdorff (LCH) spaces, namely topological spaces \( S \) such that every \( x \in S \) has a compact neighbourhood. In that case, the relevant algebra is the set of functions that vanish at infinity, namely those \( f \in C_\mathbb{R}(S) \) such that \( \forall \epsilon > 0 \), the set \( \{ x \in S : |f(x)| \geq \epsilon \} \) is compact. Indeed, it suffices to apply the above to the one-point compactification \( X = S \cup \{ \infty \} \) of \( S \), noting that every continuous function on \( S \) vanishing at infinity has a continuous extension to \( X \) (see Sheet 2, Problem 2).

We conclude this chapter with Urysohn’s lemma. It is again about separating sets, but now using continuous functions. Both the lemma and the following proposition upon which its proof lies can be phrased very explicitly in the context of metric spaces. Here, we present the proofs for a more general locally compact Hausdorff space. First of all,

**Proposition 1.22.** Let \( S \) be a LCH space. Let \( K \subset U \subset S \), where \( K \) is compact and \( U \) is open. There is an open set \( O \) with compact closure such that

\[
K \subset O \subset \overline{O} \subset U.
\]

**Proof.** Since \( S \) is LCH, every point of \( K \) has an open neighbourhood with compact closure. Since \( K \) is compact, there is finite subcover of such neighbourhoods. Hence \( K \) is a subset of their union \( V \) which has a compact closure (indeed, \( \overline{V} \) is the finite union of the compact closures of the neighbourhoods). If \( U = S \), then \( O = V \) satisfies the conclusion of the theorem. Otherwise, the complement \( U^c \) is nonempty. By the Hausdorff property, for any \( x \in U^c \subset K^c \), there is an open set \( O_x \) such that \( K \subset O_x \) and \( x \notin \overline{O_x} \), see Lemma 1.17.
It follows that
\[ \bigcap_{x \in U^c} U^c \cap V \cap \overline{O_x} = \emptyset, \]
where each \( U^c \cap V \cap \overline{O_x} \) is a compact subset of \( V \), hence closed. By the finite intersection property, there are finitely many \( \{x_1, \ldots, x_n\} \) such that
\[ U^c \cap V \cap \overline{O_{x_1}} \cap \cdots \cap \overline{O_{x_n}} = \emptyset \]
and the set \( O = V \cap O_{x_1} \cap \cdots \cap O_{x_n} \supset K \) satisfies the conclusions of the theorem since
\( \overline{O} \subset V \cap \overline{O_{x_1}} \cap \cdots \cap \overline{O_{x_n}} \subset U \) and \( \overline{O} \) is compact as a closed subset of a compact set. \( \square \)

We recall that the support of a complex-valued function \( f \) is given by
\[ \text{supp}(f) = \{x \in S : f(x) \neq 0\}. \]

We denote by \( C_c(S) \) the set of compactly supported continuous functions on \( S \). With these definitions, we denote
\[ K \prec f \]
for a compact set \( K \) and a \( f \in C_c(S) \) such that \( 0 \leq f(x) \leq 1 \) for all \( x \in S \) and that \( f(x) = 1 \) for all \( x \in K \). We further denote
\[ f \prec U \]
for an open set \( U \) and a \( f \in C_c(S) \) such that \( 0 \leq f(x) \leq 1 \) for all \( x \in S \) and \( \text{supp}(f) \subset U \).

In these notations, Urysohn’s Lemma reads:

**Lemma 1.23.** Let \( S \) be a LCH space, \( K \subset U \subset S \) be respectively compact and open.

There exists a \( f \in C_c(S) \) such that
\[ K \prec f \prec U. \]
Proof. A inductive application of Proposition 1.22 yields a family of open set \( \{O_r : r \in \mathbb{Q} \cap [0,1]\} \) with compact closures such that

\[
K \subset O_1, \quad \overline{O}_0 \subset U
\]

and

\[
\overline{O}_s \subset O_r \quad \text{whenever} \quad s > r.
\]

Let

\[
f_r(x) = \begin{cases} 
 r & \text{if } x \in O_r \\
 0 & \text{otherwise}
\end{cases}, \quad g_s(x) = \begin{cases} 
 1 & \text{if } x \in \overline{O}_s \\
 s & \text{otherwise}
\end{cases}
\]

namely \( f_r = r\chi_{O_r} \) and \( g_s = s + (1-s)\chi_{\overline{O}_s} \), and

\[
f(x) = \sup \{ f_r(x) : r \in \mathbb{Q} \cap [0,1] \}, \quad g(x) = \inf \{ g_s(x) : s \in \mathbb{Q} \cap [0,1] \}.
\]

Since \( f_r \) is proportional to the characteristic function of the open set \( O_r \), it is lower semicontinuous and \( f \) being the supremum thereof, it is again lower semicontinuous (namely \( \{ x : f(x) > a \} \) is open for all \( a \in \mathbb{R} \)). Similarly \( g \) is upper semicontinuous (namely \( \{ x : g(x) < a \} \) is open for all \( a \in \mathbb{R} \)). Moreover, \( 0 \leq f \leq 1, f(x) = 1 \) for all \( x \in K \subset O_1 \), and \( \text{supp} f \subset \overline{O}_0 \subset U \). Hence, the proof is complete if we prove continuity by showing that \( f = g \). We first note that \( f_r(x) > g_s(x) \) if \( r > s \) and \( x \in O_r, x \notin \overline{O}_s \). But \( r > s \) implies \( O_r \subset O_s \), which is a contradiction. Hence \( f_r \leq g_s \) for all \( r, s \) and hence \( f \leq g \).

Finally, assume that there exists \( x \) such that \( f(x) < g(x) \). There are \( r, s \in \mathbb{Q} \) such that \( f(x) < r < s < g(x) \). The first inequality implies that \( x \notin O_r \) while the third inequality implies that \( x \in \overline{O}_s \), and both together are in contradiction with the second inequality. Hence \( f = g \). \( \square \)

We conclude with two useful consequences of the lemma.
Proposition 1.24. Let \((S, \mathcal{T})\) be a LCH space, let \(K\) be compact and let \(\{O_i : 1 \leq i \leq n\}\) be a finite open cover of \(K\). There exists functions \(\{f_i \in C_c(S) : 1 \leq i \leq n\}\) such that

(i) \(\sum_{i=1}^{n} f_i(x) = 1\) for all \(x \in K\)

(ii) \(f_i \prec O_i\) for all \(1 \leq i \leq n\)

The family \(\{f_i : 1 \leq i \leq n\}\) is called a \textit{partition of unity} on \(K\) that is subordinate to \(\{O_i : 1 \leq i \leq n\}\).

\textit{Proof.} Let \(x \in K\). By assumptions, there are \(i_x\) such that \(x \in O_{i_x}\). Moreover, \(\{x\}\) is a compact, hence there is a compact neighbourhood \(x \in N_x \subset O_{i_x}\) by Proposition 1.22.

By compactness, there are \(x_1, \ldots, x_m \in K\) such that \(K \subset \bigcup_{j=1}^{m} N_{x_j} \subset \bigcup_{j=1}^{m} N_{x_j} \). For \(1 \leq i \leq n\), let \(K_i = \bigcup_j N_{x_{ij}}\) where \(N_{x_{ij}} \subset O_i\). Then \(K_i\) is compact and \(K_i \subset O_i\), so that there is a compactly supported continuous \(g_i\) such that \(K_i \prec g_i \prec O_i\) by Urysohn’s lemma.

Since \(K \subset \bigcup_{i=1}^{n} K_i\), we have that \(\sum_{i=1}^{n} g_i \geq 1\) on \(K\). Now \(W = \{x : \sum_{i=1}^{n} g_i(x) > 0\}\) is open (as the preimage of an open set by a continuous function) so that by Urysohn’s lemma again, there is \(f\) such that \(K \prec f \prec W\). Let \(g_{n+1} = 1 - f\). Then by construction \(\sum_{i=1}^{n+1} g_i > 0\), so that \(f_i = g_i/\sum_{j=1}^{n+1} g_j\) is well-defined on \(S\) for \(1 \leq i \leq n\). Clearly, \(\text{supp}(f_i) = \text{supp}(g_i) \subset O_i\). Finally, \(g_{n+1} = 0\) on \(K\) implies that \(\sum_{i=1}^{n} f_i = 1\). \(\square\)

Proposition 1.25 (Tietze’s extension). Let \((S, \mathcal{T})\) be a LCH space, let \(K\) be compact and let \(f \in C(K)\). There exists \(F \in C_c(S)\) such that \(F(x) = f(x)\) for all \(x \in K\).

\textit{Proof.} Since \(f\) is continuous on a compact space, it is bounded and we assume without loss that \(-1 \leq f \leq 1\) on \(K\). Let \(V\) be as in the proof of Urysohn’s lemma be open with compact closure and such that \(K \subset V\). The sets \(K^+ = \{x \in K : f(x) \geq 1/3\}\) are disjoint closed subsets of \(K\) and hence compact. Applying Urysohn’s lemma first to \(K^+\) and \(V \setminus K^-\), second to \(K^-\) and \(V \setminus K^+\), taking the difference and rescaling, there is a
function $f_1 \in C_c(S)$ such that $f_1 = 1/3$ on $K^+$, $f_1 = -1/3$ on $K^-$, and $-1/3 \leq f_1 \leq 1/3$ and $\text{supp}(f_1) \subset V$. Hence $-2/3 \leq f - f_1 \leq 2/3$ on $K$. We repeat this with $f - f_1$ replacing $f$ to obtain $f_2 \in C_c(S)$ with $\text{supp}(f_2) \subset V$, such that $|f_2| \leq (1/3)(2/3)$ on $S$ and $|f - f_1 - f_2| \leq (2/3)^2$ on $K$. This procedure provides a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_c(S)$ such that $|f_n| \leq (1/3)(2/3)^{n-1}$ on $S$ and $|f - \sum_{j=1}^{n} f_j| \leq (2/3)^n$ on $K$. This shows that the series $F = \sum_{j=1}^{\infty} f_j$ converges uniformly on $S$, hence $F$ is continuous, and it converges to $f$ on $K$. Moreover, $\text{supp}(F) \subset V$. \[\square\]
2. Normed vector spaces

**Definition 2.1.** A normed linear space \((V, \| \cdot \|)\) is a vector space \(V\) over \(\mathbb{C}\) (or \(\mathbb{R}\)) equipped with a norm \(\| \cdot \| : V \to [0, \infty)\) such that

(i) \(\|v\| \geq 0\) for all \(v \in V\) and \(\|v\| = 0 \iff v = 0\),

(ii) \(\|\lambda v\| = |\lambda|\|v\|\) for all \(v \in V, \lambda \in \mathbb{C}\),

(iii) \(\|v + w\| \leq \|v\| + \|w\|\) for all \(v, w \in V\) (Minkowski’s inequality).

Functional analysis is often interested in mappings between normed linear spaces. An important and simple class is that of bounded linear transformations.

**Definition 2.2.** Let \((V_1, \| \cdot \|_1), (V_2, \| \cdot \|_2)\) be two normed linear spaces. A bounded linear transformation is a function \(T : V_1 \to V_2\) such that

(i) \(T(\lambda v + w) = \lambda T(v) + T(w)\) for all \(v, w \in V_1, \lambda \in \mathbb{C}\)

(ii) There exists \(C \geq 0\) such that \(\|Tv\|_2 \leq C\|v\|_1\) for all \(v \in V_1\)

The norm of \(T\) is the smallest such constant, namely

\[
\|T\| = \sup \left\{ \frac{\|Tv\|_2}{\|v\|_1} : v \in V_1, v \neq 0 \right\}.
\]

The set of all bounded linear transformations is a vector space denoted \(\mathcal{L}(V_1, V_2)\), and the norm just defined is referred to as the operator norm. We briefly check that the triangle inequality holds:

\[
\|M + T\| \leq \sup \left\{ \frac{\|Mv\|_2 + \|Tv\|_2}{\|v\|_1} : v \in V_1, v \neq 0 \right\}
\]

\[
\leq \sup \left\{ \frac{\|Mv\|_2}{\|v\|_1} : v \in V_1, v \neq 0 \right\} + \sup \left\{ \frac{\|Tv\|_2}{\|v\|_1} : v \in V_1, v \neq 0 \right\}
\]

\[
= \|M\| + \|T\|,
\]
by the triangle inequality of the norm $\| \cdot \|_2$ and the property of the supremum.

Any normed linear space $(V, \| \cdot \|)$ is a metric space, with the metric being

$$d(v, w) = \| v - w \|.$$ 

If not otherwise stated, the topology on a normed linear space is always the one induced by the norm.

Interestingly, linearity implies that boundedness and continuity are equivalent:

**Proposition 2.3.** Let $T : V_1 \to V_2$ be a linear transformation between two normed linear spaces $(V_1, \| \cdot \|_1), (V_2, \| \cdot \|_2)$. The following are equivalent:

(i) $T$ is continuous at $v_0 \in V_1$

(ii) $T$ is continuous everywhere

(iii) $T$ is bounded

*Proof.* (ii)$\Rightarrow$(i) is trivial. If (i) holds, there is $r > 0$ such that $\| v - v_0 \|_1 < 2r^{-1}$ implies $\| Tv - Tv_0 \|_2 < 1$. For any $w \in V_1$, the vector $v = \frac{w}{r\|w\|_1} + v_0$ is such that $\| v - v_0 \|_1 = r^{-1}$ and so

$$\| Tv \|_2 = r\|w\|_1 \| T(v - v_0) \|_2 = r\|w\|_1 \| Tv - Tv_0 \|_2 \leq r\|w\|_1,$$

which is (iii). Finally, assuming (iii), $\|Tv_1 - Tv_2\|_2 = \|T(v_1 - v_2)\|_2 \leq r\|v_1 - v_2\|_1$, so that (iii) implies (ii). \qed

In $\mathbb{R}^n$, the closed unit ball is compact. Interestingly, this fact turns out to be characteristic of finite-dimensional normed linear spaces:

**Theorem 2.4.** Let $V$ be an infinite-dimensional normed linear space. Then the set $B_1 = \{ v \in V : \|v\| \leq 1 \}$ is not compact.
Proof. We construct a sequence \((w_n)_{n \in \mathbb{N}}\) in \(\mathcal{B}_1\) recursively as follows. Let \(w_1 \in \mathcal{B}_1\) be arbitrary. Given \(\{w_1, \ldots, w_n\}\), let \(W_n\) be their span, which is finite-dimensional and hence closed. Since \(V\) is infinite-dimensional, \(V \setminus W_n \neq \emptyset\). We claim that there exists \(w_{n+1} \in V\) such that
\[
\|w_{n+1}\| = 1, \quad \|w_{n+1} - w\| > \frac{1}{2} \quad (w \in W_n).
\]
It follows that \(\|w_{j'} - w_j\| > 1/2\) for all \(j, j' \in \mathbb{N}\) so that the sequence \((w_n)_{n \in \mathbb{N}}\) in \(\mathcal{B}_1\) has no convergent subsequence. Let \(x \in V \setminus W_n\). Since \(W_n\) is closed, \(\delta_0 = \inf\{\|x - w\| : w \in W_n\} > 0\). In particular, there is \(w_0 \in W_n\) such that \(\|x - w_0\| < 2\delta_0\). We let \(w_{n+1} = \frac{x - w_0}{\|x - w_0\|}\), and note that \(\|w_{n+1}\| = 1\) and that
\[
\inf_{w \in W_n} \|w_{n+1} - w\| = \inf_{w \in W_n} \frac{\|x - w_0 - w\|}{\|x - w_0\|} = \frac{\inf_{w \in W_n} \|x - w\|}{\|x - w_0\|} > \frac{1}{2},
\]
where we simply renamed \(w \|x - w_0\| \to w\) in the first equality and similarly \(w - w_0 \to w\) in the second, since \(W_n\) is a linear space. \(\square\)

Here is one of the most important definitions of the course:

**Definition 2.5.** A Banach space is a complete normed linear space.

Recall that a normed vector space is complete if every Cauchy sequence is convergent.