MATH 421/510, 2019WT2

Homework set 8 – due March 13

Problem 1. Let $1 < p < \infty$ and let ℓ^p be the space of complex sequences such that $\sum_{j=1}^{\infty} |z_n|^p < \infty$. Note that $\ell^q \simeq (\ell^p)^*$ whenever (p, q) are dual indices (why?). Let $(z^n)_{n\in\mathbb{N}}$ be a sequence in ℓ^p. Prove that $z^n \rightharpoonup z$ if and only if $(z^n)_{n\in\mathbb{N}}$ is bounded and $z_j^n \to z_j$ as $n \to \infty$ for all $j \in \mathbb{N}$.

Problem 2. Let V be a real normed vector space. Let $(v_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence to v in V. Prove *Mazur's theorem*: There is a sequence $(w_j)_{j\in\mathbb{N}}$ in V such that $w_j \to v$ strongly and each w_j is a finite convex combination of $\{v_n : n \in \mathbb{N}\}$, namely

$$w_j = \sum_{n=1}^{N_j} \alpha_n^j v_n, \qquad \alpha_n^j \geq 0, \sum_{n=1}^{N_j} \alpha_n^j = 1.$$

Problem 3. Let V be a real normed vector space and $X \subset V$ be a closed convex subset. Let $F : X \to \mathbb{R}$ be strongly continuous and convex. Prove that F is weakly sequentially lower semicontinuous.
Hint: Use Problem 2.

Problem 4. Let $0 < \epsilon < 1$ and let $T_\epsilon \in L^\infty((0, 1))^*$ be defined by

$$T_\epsilon f = \frac{1}{\epsilon} \int_0^\epsilon f dx.$$

Prove that $\{T_\epsilon : 0 < \epsilon < 1\}$ is bounded in $L^\infty((0, 1))^*$, but that it is not weakly-* sequentially compact. Interpret your result in view of the Banach-Alaoglu theorem.
Hint: Consider the function $f = \sum_{n=1}^{\infty} (-1)^n \chi_{[\epsilon_{n+1},\epsilon_n)}(x)$.

Problem 5. Let $1 < p \leq 2$ and $f, g \in L^p(\Omega)$. Use *Hanner's inequality*

$$2^p(\|f\|_p^p + \|g\|_p^p) \geq (\|f + g\|_p + \|f - g\|_p)^p + \big|\|f + g\|_p - \|f - g\|_p^p\big|^p$$

to show that if $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\|f_n\|_p \to \|f\|_p$, then $\|f_n - f\|_p \to 0$ as $n \to \infty$.