Problem 1. This is an optional exercise. It will not be graded.
Let $V$ be a real normed vector space, and let $A, B$ be non empty, disjoint and convex subsets of $V$. Assume that $A$ is open.

(i) Let $a_0 \in A, b_0 \in B$ and $x_0 = b_0 - a_0$. Let $C = A - B + x_0 = \{ a - b + x_0 : a \in A, b \in B \}$. Prove that $C$ is convex, open and $0 \in C, x_0 \notin C$.

(ii) Define the Minkowski functional as the map $p : V \to \mathbb{R}$

\[ p(x) = \inf \{ \lambda > 0 : x \in \lambda C \}. \]

Prove there is $M > 0$ such that $p(x) \leq M \| x \|$ and that $C \subset \{ x \in V : p(x) < 1 \}$.

(iii) Prove that $p$ is convex. Hint. Show first that $p(\alpha x) = \alpha p(x)$ for $\alpha > 0$.

(iv) Prove that there is $\ell \in V^*$ and $\lambda \in \mathbb{R}$ such that

\[ \ell(a) < \lambda \leq \ell(b) \]

for all $a \in A, b \in B$. Hint. Let $f : \text{span}\{x_0\} \to \mathbb{R}$ be defined by $f(tx_0) = t$. Use Hahn-Banach.

Assume now that $A$ is compact and $B$ is closed.

(v) Prove that there is $\ell \in V^*$ and $\lambda \in \mathbb{R}$ such that

\[ \sup \{ \ell(a) : a \in A \} < \lambda < \inf \{ \ell(b) : b \in B \}. \]

In other words, the convex sets $A, B$ can be separated by the hyperplane $\{ x \in V : \ell(x) = \lambda \}$.

Problem 2. Let $V$ be a vector space and let $\| \cdot \|_1, \| \cdot \|_2$ be two norms on $V$ such that $\| v \|_1 \leq c \| v \|_2$ for all $v \in V$. Prove that if $V$ is complete with respect to both norms, then they are equivalent.

Problem 3. Let $V, W$ be two Banach spaces with norms $\| \cdot \|_V, \| \cdot \|_W$. Let $T \in \mathcal{L}(V, W)$ be such that $\text{Ran}(T)$ is closed and $\dim \text{Ker}(T) < \infty$. Let $\| \cdot \|$ denote another norm on $V$ such that $\| x \| \leq M \| x \|_V$ for all $x \in V$. Prove that there exists $C > 0$ such that

\[ \| x \|_V \leq C(\| Tx \|_W + \| x \|) \]

for all $x \in V$. Hint. Argue by contradiction.

Problem 4. Let $V = \{ z \in \ell^1 : \sum_{n=1}^{\infty} n |z_n| < \infty \}$.

(i) Prove that $V$ is a proper dense subspace of $\ell^1$

(ii) Let $T : V \to \ell^1$ be defined by $(Tz)_n = nz_n$. Prove that $T$ is unbounded and closed.

(iii) Prove that $S = T^{-1} : \ell^1 \to V$ is bounded and surjective but not open.

Problem 5. Let $V, W$ be Banach spaces and let $D \subset V$ be a dense subset. Let $T : D \to W$ be a bounded linear transformation. Prove that there is a unique extension $\tilde{T} : V \to W$ such that $\| \tilde{T} \|_{\mathcal{L}(V, W)} = \| T \|_{\mathcal{L}(D, W)}$

Problem 6. Let $V$ be an infinite-dimensional normed linear space.

(i) Let $\ell_1, \ldots, \ell_n \in V^*$. Prove that there is $v_0 \in V, v_0 \neq 0$ such that $\ell_j(v_0) = 0$ for $1 \leq j \leq n$.

(ii) Show that the weak closure of the unit sphere $\{ v \in V : \| v \| = 1 \}$ is the closed unit ball $\{ v \in V : \| v \| \leq 1 \}$.

(iii) Show that the open unit ball $\{ v \in V : \| v \| < 1 \}$ is not weakly open.