MATH 421/510, 2019WT2

Homework set 6 – due February 28

Problem 1. Consider the Banach spaces of convergent sequences C, C_0 defined on sheet 5, and denote the norm defined there by $\|z\|_\infty$. Moreover, let ℓ^1 be the space of absolutely convergent sequences. It is a normed vector space equipped with the norm $\|z\|_1 = \sum_{n=1}^\infty |z_n|$.
(i) Prove that $(C_0)^* \simeq \ell^1$
(ii) Characterize the dual of C.
Hint. $z \mapsto \lim_{n\to\infty} z_n$ is a bounded linear functional on C that is zero on C_0.

Problem 2. Let X be a Banach space and $A \subset X$. Let

$$A^\perp = \{F \in X^* : F(x) = 0 \text{ for all } x \in A\}.$$

(i) Show that $A^\perp$ is a closed linear subspace of X^*.
(ii) Let now A be a linear subspace, and let $J : X^*/A^\perp \to A^*$ by

$$J([F]) = F\restriction_A$$

where $F\restriction_A$ is the restriction of F to A. Show that J is a well-defined isometric isomorphism.

Problem 3. Let V, W be normed vector spaces and let $T \in \mathcal{L}(V, W)$. The *adjoint* $T^\times : W^* \to V^*$ of T is the map defined by

$$(T^\times g)(x) = g(T(x)) \qquad x \in V.$$

(i) Prove that $T^\times \in \mathcal{L}(W^*, V^*)$ such that $\|T^\times\| = \|T\|$.
(ii) Prove that $\mathrm{Ker}(T^\times) = \mathrm{Ran}(T)^\perp$.
(iii) Prove that $\mathrm{Ker}(T) = {}^\perp(\mathrm{Ran}(T^\times))$.
Remark. Here, ${}^\perp S = \{x \in V : \ell(x) = 0 \text{ for all } \ell \in S\}$.
(iv) Prove that if $S \in \mathcal{L}(W, X)$, then $(ST)^\times = T^\times S^\times$ and that if T is invertible, then so is $T^\times$ with $(T^\times)^{-1} = (T^{-1})^\times$.

Problem 4. Let V be a normed vector space, $W \subset V$ a subspace and let $v \in V \setminus W$. Let $d = \inf\{\|w - v\| : w \in W\}$ be the distance between v and W. Prove that there is $\ell \in V^*$ such that $\|\ell\|_{V^*} \leq 1$, $\ell(v) = d$ and $\ell(w) = 0$ for all $w \in W$.