MATH 421/510, 2019WT2

# Homework set 5 – due February 14

**Problem 1.** Show that the space $C$ of all convergent sequences of complex numbers and the space $C_0$ of all sequences that converge to zero are Banach spaces with respect to the norm

$$\|(z_n)_{n\in\mathbb{N}}\| = \sup\{|z_n| : n \in \mathbb{N}\}.$$

**Problem 2.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued measurable functions on $\Omega$.
(i) Assume that $\|f_n\|_\infty < \infty$ for all $n \in \mathbb{N}$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges w.r.t. $\|\cdot\|_\infty$ iff there is a set $E$ of measure zero such that $f_n \to f$ uniformly on $\Omega \setminus E$.
(ii) Assume now that $\mu$ is a finite measure. Assume that $f_n(x) \to f(x)$, $\mu$-almost everywhere. Let $\epsilon > 0$. Prove that there is $R_\epsilon$ with $\mu(R_\epsilon) > \mu(\Omega) - \epsilon$ such that $f_n \to f$ uniformly on $R_\epsilon$.

**Problem 3.** (i) Let $K$ be a measurable, complex-valued function on $(0,\infty)\times(0,\infty)$ such that $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ for any $\lambda > 0$. Assume that there is $1 \le p \le \infty$ such that $\int_0^\infty |K(x,1)|x^{-1/p} = C < \infty$. Prove that for any $f \in L^p((0,\infty))$

$$(Tf)(y) = \int_0^\infty K(x,y)f(x)dx < \infty$$

defines a bounded linear transformation from $L^p((0,\infty))$ to itself with $\|T\| \le C$.
(ii) Let $1 \le p < \infty$ and $r > 0$. Use (i) to prove that for any measurable function $h : (0,\infty) \to [0,\infty)$,

$$\int_0^\infty \frac{1}{y^{1+r}}\Big(\int_0^y h(x)dx\Big)^p dy \le \Big(\frac{p}{r}\Big)^p \int_0^\infty \frac{1}{x^{1+r-p}} h(x)^p dx.$$

**Problem 4.** Let $1 < p < \infty$ and let $f, g \in L^p(\Omega)$. Prove that $\mathbb{R} \ni t \mapsto N(t) = \|f + tg\|_p^p$ is differentiable and that

$$N'(0) = \frac{p}{2}\int_\Omega |f|^{p-2}\left(\bar{f}g + f\bar{g}\right)d\mu.$$