MATH 421/510, 2019WT2

Homework set 2 – due January 24

Problem 1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces.
(i) Prove that if $f : X \to Y$ is continuous then $\lim_{n\to\infty} x_n = x \;\Rightarrow\; \lim_{n\to\infty} f(x_n) = f(x)$.
(ii) Assume that X is first countable. Prove that $\lim_{n\to\infty} x_n = x \;\Rightarrow\; \lim_{n\to\infty} f(x_n) = f(x)$ implies $f : X \to Y$ is continuous.

Problem 2. Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be topological spaces.
(i) Prove that if S_1 is connected and $f : S_1 \to S_2$ is continuous, then $f(S_1)$ is connected.
(ii) The set S_1 is called *arcwise connected* if for each $x, y \in S_1$, there exists a continuous function $f : [0,1] \to S_1$ such that $f(0) = x$ and $f(1) = y$. Prove that every arcwise connected space is connected.
(iii) Let $S = \{(s,t) \in \mathbb{R}^2 : s \neq 0,\ t = \sin(1/s)\} \cup \{(0,0)\}$ equipped with the relative topology inherited from $\mathbb{R}^2$. Prove that S is connected but not arcwise connected.

Problem 3. Let S be a Hausdorff space that is not compact. The *one-point compactification* of S is $X = S \cup \{\infty\}$ with ∞ being a single point and with the collection $\mathcal{T}$ of open sets for X being defined as follows. Let $Y \subset X$. If $\infty \notin Y$, then $Y \in \mathcal{T}$ if and only if Y is an open subset of S. If $\infty \in Y$, then $Y \in \mathcal{T}$ if and only if $X \setminus Y$ is a compact subset of S.
(i) Prove that $\mathcal{T}$ is a topology.
(ii) Prove that S is dense in X.
(iii) Prove that X is compact.
(iv) Prove that, if S is locally compact, then X is Hausdorff.
(v) Let $f : S \to \mathbb{R}$ be continuous. Find a necessary and sufficient condition on f under which it has a continuous extension to X. *Hint:* Define $\lim_{x\to\infty} f(x)$

Problem 4. (i) Let X, Y be topological spaces with Y Hausdorff. Let $f, g : X \to Y$ be continuous. Prove that if f and g agree on a dense subset of X then they agree on all of X.
(ii) Let Y be equipped with the trivial topology $\mathcal{T} = \{\emptyset, Y\}$. Prove that any function $\mathbb{R} \to Y$ is continuous.
(iii) Use (ii) with $Y = \{0,1\}$ to construct a counterexample to the statement of part (i). Explain why this is possible.

Remark. 'Prove that $(S, \mathcal{T})$ is compact if and only if S has the finite intersection property.' is a short exercise in Boole-Morgan's laws.