Problem 1. Let $A : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ obey

$$\sup_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} |A(n, m)| < \infty \quad \text{and} \quad \sup_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} |A(n, m)| < \infty.$$ 

Prove that $(A)_n = \sum_{m \in \mathbb{N}} A(n, m)b_m$ defines a bounded linear operator on $\ell^2$ and give a bound on its norm.

Problem 2. Let $\mathcal{H} = L^2(X, \mu)$. Assume that for any measurable $A \subset X$ with $\mu(A) = \infty$, there is $B \subset A$ such that $0 < \mu(B) < \infty$. Let $f : X \to \mathbb{C}$ be a bounded measurable function. Let $T_f$ be the operator on $\mathcal{H}$ defined by $(T_f g)(x) = f(x)g(x)$. Hint for both (i,ii). Use characteristic functions.

(i) Prove that $\lambda \in \sigma(T_f)$ iff $\mu\{x \in X : |f(x) - \lambda| < \epsilon\} > 0$ for all $\epsilon > 0$.

(ii) Prove that $\lambda$ is an eigenvalue of $T_f$, namely there is $g_\lambda \in \mathcal{H}$ such that $T_f g_\lambda = \lambda g_\lambda$ iff $\mu\{x \in X : f(x) = \lambda\} > 0$.

(iii) Determine the spectrum and the set of eigenvalues in the case $X = (0, 1)$ with Lebesgue measure and $f(x) = x$.

Problem 3. Let $f(\theta) \in L^2([0, 2\pi])$. For $n \in \mathbb{Z}$, let $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ and $c_n = \langle \varphi_n, f \rangle$.

(i) Prove that $\lim_{n \to \infty} c_n = 0$.

(ii) Let $S_N(\theta) = \sum_{n=-N}^{N} c_n \varphi_n(\theta)$. Extend $f$ to a periodic function on $\mathbb{R}$ and prove that

$$S_N(\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} f(x + \theta) \frac{\sin(N + 1/2)x}{\sin \frac{x}{2}} \, dx$$

Problem 4. Assume that $f$ is periodic of period $2\pi$ and continuously differentiable. Define $\varphi_n, c_n$ and $S_N$ as in Problem 2.

(i) Prove that $\lim_{N \to \infty} S_N(\theta) = f(\theta)$. Hint. Prove that the integral of the kernel in $S_N$ equals $1$.

(ii) Let $b_n = \langle \varphi_n, f' \rangle$; prove that $\sum_{n=-N}^{N} |b_n|^2$ and $\sum_{n=-N}^{N} n^2 |c_n|^2$ are convergent as $N \to \infty$.

(iii) Prove that $\sum_{n=-N}^{N} |c_n|$ is convergent as $N \to \infty$.

(iv) Prove that $S_N(\theta)$ converges uniformly to $f(\theta)$. 