Problem 1. Let $V$ be a Banach space and let $T \in \mathcal{L}(V)$. Let $\Omega \supset \sigma(T)$ and let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be analytic. Let $\gamma$ be a positively oriented simple closed $C^2$-path in $\Omega \cap \rho(T)$ whose interior contains $\sigma(T)$. Define

$$f(T) = \frac{1}{2\pi i} \oint_{\gamma} (z1 - T)^{-1} f(z) dz.$$ 

(i) Let $P(z) = \sum_{j=1}^{N} a_j z^j$ be a polynomial. Prove that $P(T) = \sum_{j=1}^{N} a_j T^j$

Hint. See equation (2) in Solution 9

(ii) Prove that the (holomorphic) functional calculus $f \mapsto f(T)$ is a homomorphism between the algebra of functions analytic in $\Omega$ into the algebra $\mathcal{L}(V)$.

Hint. Express $(z1 - T)^{-1}(w1 - T)^{-1}$ as a difference. You are allowed to commute integrals without justification. Recall that if $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic, then $(2\pi i)^{-1} \oint_{\gamma} \frac{f(z)}{z-w} dz$ equals $f(w)$, respectively 0, if $w$ is inside, respectively outside, of the contour $\gamma$.

(iii) Show that $\sigma(f(T)) = f(\sigma(T))$.

Problem 2. (i) Let $\|v\| = \langle v, v \rangle^{1/2}$ be a Hilbert space norm. Prove that it obeys the parallelogram law

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

(ii) Let $V$ be a complex vector space. Let $\| \cdot \|$ be a norm on $V$ obeying the parallelogram law. Prove that

$$\langle u, v \rangle = \frac{1}{4} \left( \|u + v\|^2 - \|u - v\|^2 - i\|u + iv\|^2 + i\|u - iv\|^2 \right)$$

is an inner product.

Hint. That $\langle v, u \rangle = \overline{\langle u, v \rangle}$, that $\langle u, iv \rangle = i\langle u, v \rangle$, and that $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ is a result of simple but tedious calculations, which do not need to be provided. In order to prove $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$, for any $\lambda \in \mathbb{C}$, prove it first for $\lambda$ having rational real and imaginary parts, then prove the Cauchy-Schwarz inequality, and use it to extend to $\lambda \in \mathbb{C}$.

Problem 3.

Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{L}(\mathcal{H})$. The adjoint of $A$ is the operator $A^*$ defined by $\langle v, A^* w \rangle = \langle Av, w \rangle$ for all $v, w \in \mathcal{H}$.

(i) Prove that $\|A\| = \|A^*\|$. 

(ii) Prove that $\sigma(A^*) = \{ \overline{\lambda} : \lambda \in \sigma(A) \}$ and that if $A$ is invertible, then $\sigma(A^{-1}) = \{ \lambda^{-1} : \lambda \in \sigma(A) \}$.

(iii) Prove that if $A$ is self-adjoint, namely $A = A^*$, then

$$\|A\| = \sup \{|\langle v, Av \rangle| / \|v\|^2 : v \in \mathcal{H} \}$$

(iv) Prove that if $A$ is normal, namely $AA^* = A^*A$ then $r(A) = \|A\|$. Hint. Show $\|A\|^2 = \|AA^*\|$.

(v) Prove that for any $B \in \mathcal{L}(\mathcal{H})$, $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.

Hint. Compute $(\lambda 1 - AB)(1 + A(\lambda 1 - BA)^{-1}B)$
Problem 4. Let $(X, \mu), (Y, \nu)$ be two measure spaces and let $k$ be a measurable function on $X \times Y$ such that

\[ \int_{X \times Y} |k(x, y)|^2 d\mu(x) d\nu(y) < \infty. \]

Prove that $K : L^2(Y, \nu) \to L^2(X, \mu)$ defined by

\[ (Kf)(x) = \int_Y k(x, y)f(y) d\nu(y) \]

is such that for any bounded sequence $(f_n)_{n \in \mathbb{N}}$, the sequence $(Kf_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Such an operator is called compact.