1. Preliminaries

Just as an equation of the form $a \cdot x - b = 0$, where $a, b$ are integers does in general not have an integer solution, and hence rationals $\mathbb{Q}$ must be introduced, the equation $x^2 + 1 = 0$ has no real solution because the square of a real number is always non-negative.

The set of complex numbers $\mathbb{C}$ is such that:

- $\mathbb{C}$ extends the reals $\mathbb{R}$.
- Any polynomial equation has at least one solution.
- $\mathbb{C}$ is equipped with an addition law and a multiplication law satisfying the "usual" rules:
  - associativity, commutativity, neutral elements for addition and multiplication, additive inverse and multiplicative inverses for all non-zero elements: $\mathbb{C}$ is a field.

Concretely, a complex number $z \in \mathbb{C}$ is an expression of the form

$$z = x + iy$$

where $x, y$ are real numbers ($x \in \mathbb{R}, y \in \mathbb{R}$) and $i$. 
(denoted \( j \) by electrical engineers) is the imaginary unit.

- **Rules:**
  
  1. **Addition:** Let \( z = x + iy \), \( w = u + iv \)
     \[ z + w = (x + u) + i(y + v) \]

  2. **Multiplication:**
     \[ z \cdot v = (xu - yv) + i(xv + yu) \]

- **Check:**
  
  1. \( 0 + i0 \), denoted \( 0 \), is the neutral element for顶层设计: \( z + 0 = z \)
  2. \( -x + i(-y) \) is the additive inverse of \( x + iy \)
  3. \( 1 + i0 \), denoted \( 1 \), is the neutral element for \( / \), \( z \cdot 1 = z \)
  4. \( \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \) is the multiplicative inverse of \( x + iy \), \( ((x, y) \neq (0, 0)) \)

- **Importantly:**

  \[ i^2 = (0 + i \cdot 1)^2 = (0 - 1) + i(0 + 0) = -1 \]

  so that

- **Note:** With \( i^2 = -1 \), addition and multiplication follow the "usual" rules of arithmetic e.g.

  \[ z \cdot w = (x + iy)(u + iv) = xu + i(xv + iy + xu + iuv) = (xu - yv) + i(xv + yu) \]

- For \( z = x + iy \), we denote:
  
  1. \( x \) = \( \text{Re}(z) \), the real part of \( z \)
  2. \( y \) = \( \text{Im}(z) \), the imaginary part of \( z \)

  \[ z = \text{Re}(z) + i\text{Im}(z) \]
The complex plane $\mathbb{C}$ is identified with $(x, y) \in \mathbb{R}^2$.

$z = x + iy \simeq (x, y)$

$x \rightarrow \text{"real axis"}$

Note: Addition of complex numbers is the same as addition of vectors in $\mathbb{R}^2$; for multiplication, see later.

More definitions:

1. The absolute value of $z = x + iy$:

   \[ |z| = \sqrt{x^2 + y^2} \]

   (i.e., the length of the vector $(x, y)$ in $\mathbb{R}^2$).

2. The complex conjugate of $z = x + iy$:

   \[ \overline{z} = x - iy \]

   (i.e., the reflection across the real axis).

Useful facts:

\[ x \pm \overline{x} = |x| \quad \text{indeed:} \quad (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 \]

\[ \frac{1}{z} \quad \text{if } z \neq 0: \quad z^{-1} = \frac{\overline{z}}{|z|^2} \]

   \[ \text{indeed:} \quad \frac{1}{|z|^2} \cdot z = \frac{z \overline{z}}{|z|^2} = 1 \]

Example:

\[ \frac{1}{1+i} = (1-i)i \]

One equation between complex numbers is two equal.
between real number: \(z = w \iff \begin{cases} \text{re}(z) = \text{re}(w) \\ \text{im}(z) = \text{im}(w) \end{cases} \)

- \(z \) is real if and only if \(z = \bar{z} \)
- \(z \) is purely imaginary if \(z = -\bar{z} \).

\[
\text{re}(z) = \frac{1}{2} (z + \bar{z}) \quad \text{and} \quad \text{im}(z) = \frac{1}{2} (z - \bar{z}).
\]

- If \( z \neq 0 \), it can be written in polar coordinates:
  
  For \( z = x + iy \), let \( x = r \cos \theta \)
  
  \( y = r \sin \theta \)

  and

  \[ z = |z| (\cos \theta + i \sin \theta) \quad \text{with} \quad \theta \in [0, 2\pi) \]

  is called the argument of \( z \).

  Example: \(\text{arg}(i) = \frac{\pi}{2} \)

  De Moivre (at this point): \(\theta = \cos \theta + i \sin \theta \)

  In particular:

  \[ e^{\pi i} + 1 = 0 \quad \text{(Euler)} \]

- Geometric interpretation of the product:

  \[
  zw = |z||w| (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = |z||w||\left((\cos \theta \cos \phi - \sin \theta \sin \phi) + i (\cos \theta \sin \phi + \sin \theta \cos \phi)\right) = |z||w| \left|\cos(\theta + \phi) + i \sin(\theta + \phi)\right|.
  \]

  \[
  -r \quad |zw| = |z||w|
  \]

  \[
  \text{arg}(zw) = \text{arg}(z) + \text{arg}(w) \quad \text{(mod } 2\pi).\n  \]
Remarks: \( e^{i\varphi} = e^{i(\varphi + \Psi)} \) (as expected)

\[ e^{i\varphi} = \cos \varphi - i \sin \varphi = \cos (-\varphi) + i \sin (-\varphi) \]

\[ = e^{-i\varphi} \]

The choice \( \arg(t) \in [0, 2\pi) \) is arbitrary. Another choice would be \( \arg(t) \in [-\pi, \pi] \).

The complex exponential is useful to derive trig identities:

1. \( \cos(\varphi + \Psi) = \Re(e^{i(\varphi + \Psi)}) \)
   \[ = \Re(e^{i\varphi} e^{i\Psi}) \]
   \[ = \Re((\cos \varphi + i \sin \varphi)(\cos \Psi + i \sin \Psi)) \]
   \[ = \cos \varphi \cos \Psi - \sin \varphi \sin \Psi \]

2. \( \sin(\varphi + \Psi) = \Im(e^{i(\varphi + \Psi)}) = \Im(e^{i\varphi} e^{i\Psi}) \)
   \[ = \sin \varphi \cos \Psi + \cos \varphi \sin \Psi \]

Now: For any \( t \in \mathbb{C} \) (not real!), one defines

\[ e^t = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \]

namely:

\[
\begin{align*}
|e^t| &= e^{\Re(t)} \\
\arg(e^t) &= \Im(t) \quad \text{(up to a multiple of } 2\pi) \\
\end{align*}
\]

A subset \( \mathcal{L} \subset \mathbb{C} \) is naturally identified with a subset of the plane, and notions such as \( \mathcal{L} \) is open, \( \mathcal{L} \) is bounded, are then the usual one.
A set $\Omega$ is bounded if there is $M > 0$ such that $z \in \Omega \Rightarrow |z| < M$.

A set $\Omega$ is open if $z \in \Omega$ implies that there is a little disk around $z$ that is in $\Omega$.

"$\Omega$ does not contain its boundary."

A open set $\Omega$ is connected if it is impossible to find two disjoint, non-empty open sets $\Omega_1, \Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$.

A connected set $\Omega$ is always pathwise connected: for any two points $z_1, z_2 \in \Omega$, there is a physical path between $z_1$ and $z_2$ that is completely in $\Omega$. 

[Diagram of various shapes demonstrating boundedness and connectedness]
2. Functions:

We shall consider functions defined on a subset \( \mathbb{D} \subseteq \mathbb{C} \) and taking complex values:

\[
 f : \mathbb{D} \rightarrow \mathbb{C} \\
 z \mapsto f(z) = u(z) + iv(z)
\]

where \( u, v \in \mathbb{C} \).

It is often useful to think of \( f \) as a transformation of \( \mathbb{D} \).

Examples:

1. \( f(z) = z + w \) : translation by \( w \)
2. \( f(z) = e^{i\theta}z \) : rotation by angle \( \theta \)
3. \( f(z) = az \) : stretching by factor \( a \)
4. The function \( f(z) = \frac{1}{z} \).
\[ f(z) = \frac{1}{z-2} \quad \text{when} \quad z = \{ z \in \mathbb{C} : |z| < 1 \} . \]

Let \( z = \frac{1}{5} \), hence \( z = \frac{1}{5} \).

\[ |\frac{1}{5}| < 1 \iff |z| > 1 \]

\[ f(z) = \{ z \in \mathbb{C} : |z| > 1 \} \]

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b) Same with \( z' = \{ z \in \mathbb{C} : |z-1| < 1 \} \)?

\[ |\frac{1}{3} - 1| < 1 \]

Let \( z = x + iy \).

Then \( \frac{1}{z} = \frac{x - iy}{x^2 + y^2} \).

\[ x \cdot \text{Re}(z) < x^2 + y^2 \]

\[ x - 2x + 1 < 0 \implies x > \frac{1}{2} \]

So that \( f(z') = \{ z \in \mathbb{C} : \text{Re}(z) > \frac{1}{2} \} \).
1) The mapping \( \varphi : z \mapsto z + \frac{1}{2} \)

2) \( z = \rho e^{i \theta} \in \mathbb{C} : |z| = 1 \) Then

\[ |z| = e^{i \theta} + e^{-i \theta} = 2 \cos \theta, \quad \theta \in [0, 2\pi) \]

\[ e^{i \theta} \quad 1 \quad e^{-i \theta} \]

3) \( z' = \zeta \in \mathbb{C} : |z' + 5| = 1 + 5 \) \( \delta \) real, \( \geq 0 \)

4) "Blasius airfoil."

5) \( z'' = z \in \mathbb{C} : |z + 5| = \ldots \) \( \delta \) non complex

"Asymmetric Blasius airfoil."
vi) Image of $\mathbb{D} = \{z \in \mathbb{C} : \text{Im}(z) > 0 \}$ under
$$f(z) = \frac{1}{-iz + \frac{1}{z}}$$

Note: $f(z) = (f_2 \circ f_1 \circ f_0)(z)$

where:
- $f_0(z) = -iz = e^{-i\pi z}$
- $f_1(z) = z + \frac{1}{z}$
- $f_2(z) = \frac{1}{z}$

Indeed, $(f_2 \circ f_1 \circ f_0)(z) = (f_2 \circ f_1)(-iz) = f_2 \left( \frac{1}{-iz + \frac{1}{z}} \right)$

- Continuity: Suppose $f$ is defined on a domain $\Omega \subset \mathbb{C}$
  - For $\lim_{t \to t_0} f(t) = f(t_0)$

Technically: For any $\varepsilon > 0$, there is $\delta > 0$ s.t.
$$|t - t_0| < \delta \implies |f(t) - f(t_0)| < \varepsilon.$$
In other words, one can find values of \( f \) that are arbitrarily close to \( f(t_0) \) by being close enough to \( t_0 \).

**Key point:** For \( f \) to be continuous at \( t_0 \), one needs 
\[
\lim_{t \to t_0} f(t) = f(t_0)
\]
for \( t \to t_0 \) in any way.

**Examples:**
1. \( f(t) = |t|^2 \) is continuous at every point in \( \mathbb{C} \).
2. \( f(t) = \frac{t^2 - 5}{t - 3} \) is continuous for \( t \neq 3 \).
3. \[
f(t) = \begin{cases} 
\frac{t^2 + 1}{t + i} & t 
eq -i \\
-2i & t = -i 
\end{cases}
\]
is continuous for every point in \( \mathbb{C} \).
4. \( f(t) = \frac{1}{|t|} \) is continuous for all \( t \neq 0 \).

Why discontinuous at \( t_0 = 0 \)?
Choose sequence \( t_n = \frac{1}{n} e^{i\theta} \) for some fixed \( \theta \in [0, 2\pi) \).

Clearly \( t_n \to 0 \) as \( n \to \infty \).

But \( f(t_n) = \frac{1}{\frac{1}{n} e^{i\theta}} = e^{i\theta} \) for all \( n \in \mathbb{N} \).

Hence \( f(t_n) \) converges but with different limits for different choices of \( \theta \).