Problem 1. (i) Let $f, g$ be holomorphic in a domain $\Omega$. Assume that $z_0 \in \Omega$ is a zero of multiplicity 1 of $g$ and that $f(z_0) \neq 0$. Show that

$$\text{Res} \left( \frac{f(z)}{g(z)} ; z_0 \right) = \frac{f(z_0)}{g'(z_0)}$$

(ii) Compute

$$\oint_{|z|=2\pi} \tan(z) \, dz$$

Problem 2. Compute

$$I = \int_{-\infty}^{+\infty} \frac{e^{2x}}{\cosh(\pi x)} \, dx$$

*Hint:* Use a rectangular closed curve with corners $R, R + i, -R + i, -R$.

Problem 3. Let $P$ be a polynomial of degree $d$, and let $\alpha$ be a closed, positively oriented simple curve such that no zero of $P$ lies on $\alpha$. Show that

$$\frac{1}{2\pi i} \oint_{\alpha} P'(z) \frac{dz}{P(z)} = N,$$

where $0 \leq N \leq d$ is an integer counting the number of zeroes of $P$ inside $\alpha$ (with multiplicity: a zero of multiplicity $m$ is counted $m$ times)

*Hint:* One can write $P(z) = c(z - z_1) \cdots (z - z_d)$; why?

Problem 4. Show that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$ by following the strategy presented in class but using $f(z) = \frac{1}{z^2}$ directly (no limit to be taken at the end of the argument). Adapt the computation to account for the fact that the integrand now has a pole of order three at $z = 0$.

*Hint:* You can use that $g(z) = (\pi z) \cot(\pi z)$ is such that $\lim_{z \to 0} g''(z) = -\frac{2\pi^2}{3}$.

Problem 5. Compute

$$I = \int_0^{2\pi} \frac{\sin^2(\varphi)}{5 + 4 \cos(\varphi)} \, d\varphi.$$