Continuity

• In the common language, a "discontinuity" refers to some sort of "jump", sometimes in time, or in space. Intuitively, a function is said to be "continuous" at a point if there is no jump there. Mathematically:

A function \( f \) is continuous at \( a \) if

\[
\lim_{{x \to a}} f(x) = f(a).
\]

If \( f \) is defined on an open interval \((a, b)\), then \( f \) is said to be continuous if it is continuous at all \( a \in (a, b) \).

• Remarks: For \( f \) to be continuous at \( a \), we need:

  (i) Existence of the limit
  (ii) \( f \) is defined at \( a \) (will value \( f(a) \))
  (iii) The value of the limit is \( f(a) \)

  (in particular \( \lim_{{x \to a^-}} f(x) = f(a) = \lim_{{x \to a^+}} f(x) \)).

• \( f \) is right continuous at \( a \) if \( \lim_{{x \to a^+}} f(x) = f(a) \);

• \( f \) is left continuous at \( a \) if \( \lim_{{x \to a^-}} f(x) = f(a) \).

Moreover, to make sense of \( f \) is continuous on the closed interval \([a, b] \).
The arithmetic of limits provides rules of continuity:

If $f$ and $g$ are continuous at $a$, then

1. $f + g$ is continuous at $a$.
2. $kf$ is continuous at $a$ for any $k \in \mathbb{R}$.
3. $f/g$ is continuous at $a$, provided $g(a) \neq 0$.

Hence:

(i) A polynomial is continuous everywhere.

(ii) A rational function is continuous in its domain.

Be careful:

$$f(x) = \begin{cases} x^2 - 2x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

Since $f'(x) = x$ whenever $x \neq 2$, $f'$ is continuous on $(-\infty, 2) \cup (2, +\infty)$
but \( \lim_{x\to 2} f(x) = 4 \neq 1 = f(2) \) so that \( f \) is discontinuous at \( x = 2 \).

Something similar: if \( f \) is continuous at \( a \), and \( g \) is continuous at \( b \), will \( g(f(b)) = a \), then:

\[
[\lim_{x \to b} g(x)] = \lim_{x \to b} f(g(x))
\]

continuity of \( f \)
continuity of \( g \)

\( f \) at \( a = g(b) \)

Hence, continuity allows for:

\[
\lim f(g(x)) = f(\lim g(x))
\]

Example: \( h(x) = \sin(\sqrt{1-x^6}) \)

\[
h(x) = 1 \cdot g(x) \quad \text{will} \quad g(x) = \sqrt{1-x^6}
\]

Since \( g \) is well-defined and continuous whenever \( 1-x^6 \geq 0 \), namely \( x \in [-1, 1] \), and \( f \) is continuous on \( \mathbb{R} \), we conclude that \( h \) is continuous on \( [-1, 1] \).

\textbf{Remark:} You can use the fact that sine, cosine, exponentials are continuous on \( \mathbb{R} \), and that roots and power are continuous in their domains.
If \( f \) is continuous on \([a, b]\) (namely, its graph can be drawn without raising the pen), then \( f \) must take all possible values between \( f(a) \) and \( f(b) \).

Mathematically:

Let \( a < b \) and let \( f \) be continuous on \([a, b]\). Then for any \( F \) between \( f(a) \) and \( f(b) \), there exists (at least one) \( c \in [a, b] \) such that \( f(c) = F \).

Why is this useful (and hence why is continuity a very useful property?)? Because it tells us that equations have solutions! Indeed, any algebraic equation can be brought to the form (for example)

\[ f(x) = 0. \]
But then: 1) \( f \) is continuous, it suffices to find one \( x_0 \in \mathbb{R} \) where \( f(x_0) \) is positive, one \( x_1 \in \mathbb{R} \) where \( f(x_1) \) is negative, to conclude that the equation has a solution between \( x_0 \) and \( x_1 \).

Example: The equation \( \sin \left( \frac{\pi x}{2} \right) = 1 - x \) has a solution in \([0,1]\).

Indeed the equation is equivalent to

\[ f(x) = 0, \quad f(x) = \sin \left( \frac{\pi x}{2} \right) + x - 1. \]

and \( f \) is continuous on \( \mathbb{R} \). By the IVT, the existence of a solution follows since

\[ f(0) = -1 < 0, \quad f(1) = 1 > 0. \]

Done \( \Box \).

Remark: The bisection method:

There is a zero between 0 and 1. Compute

\[ f \left( \frac{1}{2} \right) = 0.2 \ldots > 0. \]

Hence there is a zero between 0 and \( \frac{1}{2} \).

\[ f \left( \frac{1}{4} \right) = -0.3 \ldots < 0. \]

Hence there is a zero in \([\frac{1}{4}, \frac{1}{2}]\).

Repeat and obtain a better and better approximation of the solution.