More on series

Series: \( \sum_{j=1}^{\infty} a_j \)

We have seen the limit comparison test: Given two sequences \((a_j)_{j \in \mathbb{N}}, (b_j)_{j \in \mathbb{N}}\) such that
\[
\lim_{j \to \infty} \frac{a_j}{b_j} = L
\]

i) \( \sum b_j \) converges, then \( \sum a_j \) converges.

ii) \( L \neq 0 \) and \( \sum b_j \) diverges, then \( \sum a_j \) diverges.

Other "tests" can be derived from this comparison.

The ratio test: This works for series that can be compared to the geometric series.

Let \((a_j)_{j \in \mathbb{N}}\) be a sequence such that \(a_j \neq 0\) for all \(j \geq N\), for some (large) \(N \in \mathbb{N}\).

i) \( \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| = L < 1 \), then \( \sum_{j=1}^{\infty} a_j \) converges.

ii) \( \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| = L > 1 \), or \( \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| = +\infty \),

then \( \sum_{j=1}^{\infty} a_j \) diverges.
Be careful. The ratio test provides no information in the case $\lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| = 1$.

Indeed: for any $p$-series,

$$\lim_{j \to \infty} \frac{j^p}{(j+1)^p} = \lim_{j \to \infty} \frac{1}{\left(1 + \frac{1}{j} \right)^p} = 1$$

but we already know that $\sum_{j=1}^{\infty} \frac{1}{j^p}$ diverges if $0 < p \leq 1$ and converges if $p > 1$.

Example: Take $x \in \mathbb{R}$ and let $\alpha_j = \frac{x^j}{j!}$ ($j \in \mathbb{N}$).

Then $\lim_{j \to \infty} \left| \frac{\alpha_{j+1}}{\alpha_j} \right| = \lim_{j \to \infty} \left| \frac{x^{j+1}}{x^j (j+1)!} \right| = \lim_{j \to \infty} \frac{|x|}{j} = 0$

Hence: the series

$$\sum_{j=1}^{\infty} \frac{x^j}{j!}$$

is convergent for all $x \in \mathbb{R}$. It defines a function $x \mapsto \exp(x) = 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!}$

which is also denoted $x \mapsto e^x$. This gives a definition of the number $e$:

$$e = \sum_{j=1}^{\infty} \frac{1}{j!} + 1$$
This provides one way to approximate e, by truncating the series. For example,

\[ 1 + \sum_{j=1}^{5} \frac{1}{j!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 1.78167... \]

\[ \sum_{j=1}^{10} \frac{1}{j!} = 2.7182818... \]

So far, the criteria have been "blind" to the sign of \( a_j \). But successive terms sometimes "cancel out" each other, or partly so. An illuminating example:

\[ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots \]

is convergent! In fact,

\[ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = \log \left( \frac{1}{2} \right) = -0.69314... \]

The alternating sign is crucial: without it, we would recover the (divergent) harmonic series.

The alternating series test. Let \( (A_j)_{j \in \mathbb{N}} \) be a sequence such that:

(i) \( A_j \geq 0 \) for all \( j \geq N \) for some \( N \).
(ii) \( A_{j+1} \leq A_j \) for all \( j \geq N \).
(iii) \( \lim_{j \to \infty} A_j = 0 \).
Then the alternating series
\[ \sum_{j=1}^{\infty} (-1)^{j+1} A_j = -A_1 + A_2 - A_3 + A_4 - A_5 + \ldots \]
is convergent.

**Remark:** In other words: in the special case of an alternating series, it suffices that the absolute value of the summand converges to zero for the series to converge.

**Some definitions:** Recall that \( \sum_{j=1}^{\infty} a_j \) is called **convergent** if
\[ \lim_{N \to \infty} \sum_{j=1}^{N} a_j \text{ exists.} \]

- \( \sum_{j=1}^{\infty} a_j \) is called **absolutely convergent** if \( \sum_{j=1}^{\infty} |a_j| \) is convergent.
- If \( \sum_{j=1}^{\infty} a_j \) is convergent but \( \sum_{j=1}^{\infty} |a_j| \) is divergent, the series \( \sum_{j=1}^{\infty} a_j \) is called **conditionally convergent**.

**Example:**
\[ \sum_{j=1}^{\infty} \frac{(-2)^j}{j!} \] is absolutely convergent.
\[ \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \] is conditionally convergent.

**Remark:** If a series is absolutely convergent, then it is convergent.