Some real numbers, such as $\frac{1}{3}$, $\pi$, $\sqrt{2}$, have an infinite decimal expansion:

$$\frac{1}{3} = 0.3333\ldots = 3 \cdot \frac{1}{10} + 3 \cdot \frac{1}{100} + 3 \cdot \frac{1}{1000} + \ldots$$

$$= 3 \sum_{j=1}^{\infty} \left( \frac{1}{10} \right)^j$$

such an infinite sum is called a series.

A series is a certain type of sequence. If the case just considered, we define the sequence of partial sums

$$S_N = \sum_{j=1}^{N} \left( \frac{1}{10} \right)^j$$

A little computation yields an explicit formula:

$$\frac{1}{10}S_N - S_N = \frac{1}{10} \cdot \left( \frac{1}{10} \right)^N - \frac{1}{10} - \text{first term of } S_N$$

$$= \frac{1}{10}S_N - \frac{1}{10} \text{ last term of } \frac{1}{10} S_N \text{ (all other terms cancel out)}$$

Calling $r = \frac{1}{10}$, we conclude:
\[ S_N = r \frac{1-r^N}{1-r} \]

which makes it clear that \((S_N)_{N \in \mathbb{N}}\) is a good sequence. For \(r < 1\), \(r^N \to 0\) \((N \to \infty)\) and hence
\[ \lim_{N \to \infty} S_N = \frac{r}{1-r} \]

for the example \(r = \frac{1}{10}\), \(3 \lim_{N \to \infty} S_N = 3 \frac{\frac{1}{10}}{1-\frac{1}{10}} = 3 \cdot \frac{1}{9} = \frac{1}{3} \) indeed.

In general: Let \((a_j)_{j \in \mathbb{N}}\) be a sequence. We define the sequence of partial sums associated with \((a_j)_{j \in \mathbb{N}}\) by
\[ S_N = \sum_{j=1}^{N} a_j \]

If \((S_N)_{N \in \mathbb{N}}\) is convergent \(\lim_{N \to \infty} S_N = S\), we say that the series \(\sum_{j=1}^{\infty} a_j\) converges to \(S\) and write:
\[ \sum_{j=1}^{\infty} a_j = S \]

Otherwise, the series is divergent.
A series is "an infinite sum". It is intuitive, and generally true that it can only be convergent provided the \( n \)th term that is added, namely \( a_n \), becomes smaller and smaller.

**Theorem:** If \( \sum_{j=1}^{\infty} a_j \) is convergent, then

\[
\lim_{j \to \infty} a_j = 0.
\]

**Remarks:** Most useful: the divergence test: if

the sequence \( (a_j)_{j \in \mathbb{N}} \) does not converge to zero, then the series \( \sum a_j \) is divergent.

**Example:** \( a_j = (-1)^j \). The sequence of partial sums oscillates between \((-1)^{(\text{N odd})}\) and \(0\) \((\text{N even})\).

**Be careful:** The result goes only in one direction.

The harmonic series

\[
\sum_{j=1}^{\infty} \frac{1}{j}, \text{ namely } a_j = \frac{1}{j}
\]

satisfies \( \lim_{j \to \infty} a_j = \lim_{j \to \infty} \frac{1}{j} = 0 \), but it is a divergent series. It can be shown that
\[ S_N = \sum_{j=1}^{N} \left( \frac{1}{j} \right) \geq \log (N+1) \]

We have already seen:

1. The geometric series: \[ \sum_{j=1}^{\infty} r^j \]

   is convergent if \( |r| < 1 \), with \( \sum_{j=1}^{\infty} r^j = \frac{1}{1-r} \)

   is divergent if \( |r| \geq 1 \).

2. The harmonic series: \[ \sum_{j=1}^{\infty} \frac{1}{j} \]

   is divergent.

   This is embedded in the family of \( p \)-series:

   \[ \sum_{j=1}^{\infty} \frac{1}{j^p} \]

   is convergent if \( p > 1 \)

   divergent if \( 0 < p \leq 1 \).

Note that the problem of computing the limit of these is hard. For example, \( p = 2 \):

\[ \sum_{j=1}^{\infty} \frac{1}{j^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6} \quad \text{(Euler, 1734)} \]

Sometimes, algebraic transformations allow for some simplification. For example:
Let \( a_j = \frac{2}{j(j+1)} \). Although it is easy to convince oneself that \( \sum_{j=1}^{\infty} a_j \) is convergent (because it is close to \( \frac{1}{j} \) for large \( j \), see later), its sum needs a trick.

Indeed, \( a_j = \frac{2}{j(j+1)} = \frac{1}{j} - \frac{1}{j+2} \)

so that

\[
S_N = \sum_{j=1}^{N} a_j = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right) + \left(\frac{1}{N+1} - \frac{1}{N+2}\right) + \ldots
\]

\[
= \left(1 + \frac{1}{2}\right) - \left(\frac{1}{N+1} + \frac{1}{N+2}\right)
\]

\[\to 0 \quad \text{as} \quad (N \to \infty)\]

and hence \( \lim_{N \to \infty} S_N = \frac{3}{2} \).

This type of series is called telescopic.