Limits at infinity and sequences:

Often in science, $g(t)$ describes the time evolution of some physical quantity, and one is interested in its behavior for long times.

Ex: $v(t)$ = the velocity of a skydiver; it is increasing initially but "settles" after a certain time. Mathematically: what is the limit of $v(t)$ as $t \to \infty$?

We say $\lim_{x \to \infty} f(x) = L$ if $f(x)$ is arbitrarily close to $L$, provided $x$ is large enough.

Then $f$ is said to have a horizontal asymptote.

Remark: the case $\lim_{x \to \pm \infty} f(x) = \pm \infty$ is called a vertical asymptote.

Graphically:
Examples: (i) \( \lim_{x \to \pm \infty} \frac{\sin(x)}{x} = 0 \).

Indeed, \(-1 \leq \sin(x) \leq 1\) for all \(x\) \(\in\) \(\mathbb{R}\) implies
\[-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x} \quad (x \neq 0)\]

But \(\lim_{x \to \pm \infty} \frac{1}{x} = 0\) and the claim follows by the squeeze theorem.

(ii) \(\lim_{x \to \pm \infty} \sin(x)\) does not exist: For any \(N \geq 0\), \(\sin(x)\) takes all values in \([-1, 1]\) in \([x, \infty)\).

Remark: renaming \(y = \frac{1}{x}\) this is also \(\lim_{y \to 0} \sin(\frac{1}{y})\) which we discussed on page 3.

The arithmetic of limits at finite \(a\) extends to \(a = \pm \infty\).
For rational functions \( \frac{f(x)}{g(x)} \) where \( \lim_{x \to \pm\infty} f(x) = \pm \infty \) and \( \lim_{x \to \pm\infty} g(x) = \pm \infty \), one may need some rewriting, see recitations.

A standard one is:

\[
\lim_{x \to \pm\infty} \frac{5x^2 - 1}{-7x^2 - 2x} = \lim_{x \to \pm\infty} \frac{5x^2 (1 - \frac{1}{5x})}{-7x^2 (1 - \frac{2}{7x})}
\]

Factor out the highest power,

\[
\lim_{x \to \pm\infty} \frac{5}{7} \cdot \frac{1 - \frac{1}{5x}}{1 - \frac{2}{7x}} = \frac{5}{7}
\]

Another variation: In general, we cannot measure the velocity of our sky diver continuously. Instead, we record it only stroboscopically, say every second. We obtain a sequence of measurements:

\[
(v(0), v(1), v(2), v(3), v(4), \ldots)
\]

A sequence is a function \( \alpha : \mathbb{N} \to \mathbb{R} \), often represented as an infinite list of numbers.

\[
(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)
\]

\[
(\pi, 1, 0, \pi, 5, 10017, \ldots)
\]

Remark: There is not necessarily a formula for the \( n \)th
term of the sequence, \( a_n \), or a "pattern" that determines it.

- We say \((a_n)_{n \in \mathbb{N}}\) converges to \( A \) and write
  \[
  \lim_{n \to \infty} a_n = A \quad \text{or} \quad a_n \to A \ (n \to \infty)
  \]
  if \( a_n \) is arbitrarily close to \( A \), provided \( n \) is large enough.

  *Remark.* The arithmetic of limits known for functions continues to hold for sequences; the same techniques can be applied (e.g., limits of the form \( \frac{P(n)}{Q(n)} \) where \( P, Q \) are polynomials).

**Example:** Sequence \( a_n = \frac{2^n}{5^n + n^3} \ (n \in \mathbb{N}) \)

We massage this into,

\[
  a_n = \frac{\frac{2^n}{5^n(1 + \frac{n^3}{5^n})}}{1} = \left( \frac{2}{5} \right)^n \frac{1}{1 + \frac{n^3}{5^n}}
\]

and note that \( \lim_{n \to \infty} \frac{n^3}{5^n} = 0 \): exponential functions grow much faster than polynomials.
\[ \lim_{n \to \infty} \frac{1}{1 + \frac{n^3}{5^n}} = 1 \]

and since \[ \lim_{n \to \infty} \left( \frac{2}{5} \right)^n = 0 \] (because \( \frac{2}{5} < 1 \))

we conclude \[ \lim_{n \to \infty} \frac{2^n}{5^n + n^2} = 0. \]

A useful result (similar to the squeeze theorem)
the \textit{bounded monotone convergence theorem}

Let \((a_n)_{n \in \mathbb{N}}\) be a sequence such that
\begin{itemize}
  \item \(a_n\) is \underline{increasing} \( a_{n+1} \geq a_n \) for all \( n \in \mathbb{N} \)
  \item \(a_n\) is \underline{bounded above} \( a_n < M \) for all \( n \in \mathbb{N} \)
\end{itemize}
and some real number \( M \).

Then \((a_n)_{n \in \mathbb{N}}\) converges.

Remark: The theorem provides the existence of a limit, not its numerical value!

\( \underline{\text{Similarly, a decreasing sequence that is bounded below is convergent.}} \)