Finding extrema of functions has obvious applications in engineering and science. It is also how most of the fundamental laws of nature arise.

Example: refraction law (Snell, Descartes, Ibn Sahl)

\[ T(t) = \frac{\sqrt{t^2 + y_1^2}}{n_1 v} + \frac{\sqrt{(x-t)^2 + y_2^2}}{n_2 v} \quad t \in (0, x) \]

is differentiable on \((0, x)\), continuous on \([0, x]\)

Critical points?

\[ T'(t) = \frac{1}{v} \left( \frac{t}{n_1 \sqrt{t^2 + y_1^2}} + \frac{x-t}{n_2 \sqrt{(x-t)^2 + y_2^2}} \right) \]
Note that
\[ \frac{x - t}{\sqrt{x^2 + y_1^2}} = \sin \theta_i \quad \text{and} \quad \frac{x - t}{\sqrt{(x-t_1)^2 + y_2^2}} = \sin \theta_r \]

Hence \[ T'(t) = 0 \quad \text{if and only if} \]

\[ \frac{1}{\sin \theta_i} = \frac{1}{\sin \theta_r} \]

\[ \frac{\sin \theta_i}{\sin \theta_r} = \frac{h_2}{h_1} \quad \text{(4)} \]

Is that a local minimum? \[ T''(t) = \left( \frac{y_1^2}{h_1 (x + y_1)^3} + \frac{y_2^2}{h_2 ((x-t_1)^2 + y_2^2)^{3/2}} \right) \frac{1}{V} > 0 \]

for all \( t \in (0, x) \)

\[ T \text{ is convex and } |T| \text{ is the condition of a minimum} \]

This is just one example of a very general principle called the Principle of Least Action.