Sketching graphs

- Recall: If $f$ has an extremum at $x_0$ and if $f''(x_0)$ exists, then $f''(x_0) = 0$.
  - No local extrema are found either
    - at critical points, or
    - at singular points (where $f''(x_0)$ does not exist).

- At a maximum, $f$ goes from increasing to decreasing.
  - At a minimum, $f$ goes from decreasing to increasing.

- If $f'(x) > 0$, $x \in (a,b)$, then $f$ is increasing on $(a,b)$.
  - If $f'(x) < 0$, $x \in (a,b)$, then $f$ is decreasing on $(a,b)$.
  - If $f'(x) = 0$, $x \in (a,b)$, then $f$ is constant on $(a,b)$.

- Complement to the above:
  - Theorem: Let $f$ be defined in a neighbourhood of $x_0$.
    1. If $f''(x_0) = 0$ and $f''(x_0) > 0$, then $x_0$ is a point of local minimum.
    2. If $f''(x_0) = 0$ and $f''(x_0) < 0$, then $x_0$ is a point of local maximum.

- This leads to a new definition: Let $f$ be continuous on $[a,b]$ and so that $f''(x) > 0$ for all $x \in (a,b)$.
  - Second derivative!
Then \( f \) is \[ \text{convex (or concave up).} \]

If, on the other hand, \( f''(x) \leq 0 \) for all \( x \in (a,b) \), then \( f \) is \[ \text{concave (or concave down).} \]

Finally, if \( f''(x_0) = 0 \) and the concavity of \( f \) changes across \( x = x_0 \), then \( x_0 \) is an \[ \text{inflection point.} \]

In other words, \( f \) is convex if its slope is increasing.

\[ \text{Graphically,} \]

\[ \text{convex} \quad \text{concave.} \]

\[ x_0 \]

\[ \text{Inflection point} \]

- Rephrasing the above, if \( f \) is convex in a neighborhood of a critical point \( x_0 \), then \( x_0 \) is a local minimum. If \( f \) is concave, then \( x_0 \) is a local maximum. \[ \text{of } f \]

- Remark: Convexity is generally rephrased in the more intuitive notion that the region above the graph of the function is convex.

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The secant line is completely above the graph of \( f \).

\[ y = \lambda x + (1 - \lambda y) \quad (\lambda \in (0, 1)) \]

Conversely, \( \frac{d}{dx} \left( \lambda x + (1 - \lambda y) \right) \leq 2 f(x) + (1 - 2) f(y) \).

- What about global extrema of \( f \) on \([a, b]\)?
  - They are to be found at:
    - critical points
    - singular points
    - the boundary of the domain \([a, b]\).
- Example: Find the local and global extrema of
  \[ f(x) = 2x^\frac{5}{3} + 3x^\frac{2}{3} \]
  on \([-1, 1]\).

First:
\[ f(x) = x^\frac{2}{3} (2x + 3) \]
so that:
\( f \) is continuous on \([-1, 1]\)
\( f \) is differentiable on
\([-1, 0) \cup (0, 1]\)
Differentiability at $x = 0$:

$$\lim_{h \to 0^+} h^{-\frac{3}{2}}(2h - 3) = \lim_{h \to 0^+} \frac{3 + 2h}{h^{\frac{3}{2}}} = 1 \cdot \infty$$

So $f$ is not differentiable at $0$ indeed.

 Everywhere else: $f''(x) = \frac{10x + 6}{3x^{\frac{1}{2}}}$

- Critical Points: $x_c = -\frac{3}{5} \in [-1, 1]$
- Singular Points: $x_s = 0 \in [-1, 1]$.

We compute $f''$ on $(-1, 0) \cup (0, 1)$:

$$f''(x) = \frac{2}{9x^{\frac{1}{2}}} (10x - 3)$$

- Inflection point: $x_i = \frac{3}{10} \in [-1, 1]$.

Conclusion:

- $f$ is increasing on $[-1, -\frac{3}{5}]$ since $f'' > 0$ there.
- $x_c$ is a point of local maximum since $f''(-\frac{3}{5}) < 0$.
- $f$ is decreasing on $[-\frac{3}{5}, 0]$ since $f'' < 0$ there.
- $f$ is increasing on $(0, 1]$ since $f'' > 0$ there.
- Hence $x_s$ is a point of local minimum.
- $f$ is concave on $[-1, 0]$ since $f'' < 0$ there and on $[0, \frac{3}{10}]$. 
$f$ is convex on $[\frac{3}{10}, 1]$.

*Values of $f$:

$f(-1) = 1$, $f(-\frac{3}{5}) \approx 1.28$, $f(0) = 0$, $f(1) = 5$

$1$ is the point of global maximum.

$0$ is the point of global minimum.

Sketch:

![Graph of a function showing concavity and points of increase and decrease.](image-url)