Disclaimer: Unless otherwise stated, all functions are defined on (a subset of) \( \mathbb{R} \), and taking values in \( \mathbb{R} \).

1. **Limits**

   A classical limit: the instantaneous velocity.

   Let \( x(t) \) be the position of a bicycle along an itinerary at time \( t \).

   The average velocity between times \( t_0 \) and \( t_n \) is given by
   \[
   \bar{v}(t_n) = \frac{\text{distance travelled}}{\text{total time}} = \frac{x(t_n) - x(t_0)}{t_n - t_0}
   \]

   But what is the instantaneous velocity at time \( t_0 \)?

   We need to consider infinitesimal time differences, i.e., very small \( t_n - t_0 \):

   \[
   v = \lim_{t_n - t_0 \to 0} \frac{x(t_n) - x(t_0)}{t_n - t_0}
   \]

   Graphically:

   ![Graphical representation of instantaneous velocity](image)
What do we mean?

That \( f(x) \) becomes arbitrarily close to \( L \) if \( x \) is small enough.

**Definition:** \[ \lim_{x \to a} f(x) = L \]

If \( f(x) \) is arbitrarily close to \( L \), provided \( x \) is sufficiently close to \( a \).

Two examples:

(i) Let \( p(x) = 3x - 5 \)

Then \[ \lim_{x \to 1} p(x) = -2 \]

Indeed: Pick \( \varepsilon > 0 \) (say \( \frac{1}{100} \) or \( \frac{1}{10000} \), ...).

If \( |x - 1| < \frac{\varepsilon}{3} \), then

\[ |p(x) - (-2)| = |(3x - 5) + 2| = 3|x - 1| < \varepsilon. \]

In other words, the difference between \( p(x) \) and \(-2\) is at most \( \frac{1}{100} \) (or \( \frac{1}{10000}, \ldots \)) if \( x \) is not further than \( \frac{1}{300} \) (or \( \frac{1}{30000}, \ldots \)) from \( 1 \).
(ii) Let \( f(x) = \sin \left( \frac{1}{x} \right) \).

Then \( \lim_{x \to 0} f(x) \) does not exist. (DNE).

Indeed:

In any interval around 0, the function \( y \) takes all values in \([-1, 1]\).

Remark: The actual value of \( f \) at \( x = 0 \) is irrelevant in the definition of the limit.

Let \( g(x) = \frac{2x-4}{x^2 + x - 6} \).

This function is not defined at \( x = 2 \).

However a plot of \( g \) gives

so we see: \( \lim_{x \to 2} g(x) = \frac{2}{5} \), and indeed

\[ \lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{2}{x+3} \quad \text{(Simplifying by } (x-2) \text{)} \]

is allowed away from \( x = 2 \).
\[ \frac{2}{5} \]

- Simple arithmetic: Assume that
  \[ \lim_{x \to 2} f(x) = F, \quad \lim_{x \to 2} g(x) = G. \]

  Then:
  \[ \lim_{x \to 2} \left( \alpha f(x) + \beta g(x) \right) = \alpha F + \beta G. \]

  \[ \lim_{x \to 2} f(x) g(x) = F \cdot G. \]

  \[ \frac{f(x)}{g(x)} = \frac{F}{G}. \]

  If \( G \neq 0 \), then
  \[ \lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{F}{G}. \]

  \[ \lim_{x \to 2} \left( f(x) \right)^n = F^n \]
  \[ \lim_{x \to 2} \left( g(x) \right)^n = G^n \]
  (whenever the \( n \)th root is well-defined).

- Remarks: As noted above, it is sometimes necessary to simplify/transform the expression of \( f \) away from \( x = 2 \) to compute \( \lim_{x \to 2} f(x) \).

  It is sometimes hard to compute the value of the limit. For example, \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) holds. (we will develop methods for this later).

  Knowing that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) are the rules above allow for the computation of the limit of rational functions and similar "composite" functions.