Closed book examination Time: 150 minutes

Last Name ___________________________ First Name ___________________________

Student Number __________ Recitation Section ____ Signature _____________________

Instructions.

1. You have 150 minutes to solve the following four problems.
2. Write your name at the top of every page.
3. Keep your student’s ID visible in front of you.
4. You must fully and carefully justify all your answers.
5. If the space below the problems is not sufficient, use the back of the pages. An additional blank sheet is provided at the back of the exam.
6. Put away books, notes, laptops, cell phones and turn off your cell phones.

Good luck!

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Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator(s), and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
Problem 1. [3+4+3+3+3 points]

In this problem, you can use the formula \( \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \) for any \( |r| < 1 \).

(i) Find the limit, if it exists, of the sequence \( (a_n)_{n=1}^{\infty} \) given by \( a_n = \frac{n! \cos(2n+1)}{(n+1)!} \).

(ii) Let \( (b_n)_{n=1}^{\infty} \) be defined by

\[
b_1 = \sqrt{2}, \quad b_{n+1} = \sqrt{2b_n} \quad (n \geq 1).
\]

Prove that the sequence is convergent and compute its limit (recall that \( \sqrt{2} \approx 1.41 \)).

(iii) Assume that the series \( \sum_{n=0}^{\infty} (3 - a_n) \) converges.
Determine whether the series \( \sum_{n=0}^{\infty} 3^n a_n \) converges or diverges.

(iv) Find the sum of the series

\[
\sum_{n=0}^{\infty} \frac{12 - 4^{n+1}}{5^n}.
\]

(v) Characterize all \( x \in \mathbb{R} \) for which the series

\[
\sum_{n=0}^{\infty} (-2)^{n+1} x^n
\]

converges, and evaluate its sum.
Problem 1 (continued)
Problem 2. [3+5+3+3 points]

(i) Compute

\[
\lim_{x \to +\infty} \frac{6x^2 - x + 3}{x\sqrt{4x - 3(\sqrt{x} - 1)}}.
\]

(ii) Compute

\[
\lim_{x \to 0^-} xe^{1/x}, \quad \lim_{x \to 0^+} xe^{1/x}.
\]

Does \(\lim_{x \to 0} xe^{1/x}\) exist?

(iii) Find all vertical asymptotes of the function

\[f(x) = \frac{x}{\sin(\pi x)}\]

(iv) Let \(g(x) = x^5 + x - 6\). Compute the two limits \(\lim_{x \to \pm\infty} g(x)\),
and conclude that the equation \(g(x) = 0\) has at least one real solution.
Problem 2. (continued)
Problem 3. [3+4+3+4 points]

(i) Let \( f(x) = e^{-\frac{x^2}{2}} \). Compute \( f'(x) \) and \( f''(x) \).

(ii) The arccosine function \( \arccos : [-1,1] \rightarrow [0,\pi] \) is defined as the inverse of the cosine function:
\[
\arccos(\cos(x)) = x \quad (x \in [0,\pi]), \quad \cos(\arccos(x)) = x \quad (x \in [-1,1]).
\]
Prove that
\[
\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.
\]

(iii) Let \( g(x) = x^5 - \sqrt{x} + \cos(2\pi x) + \pi \). Prove that \( g \) has at least one critical point in \((0,1)\).

(iv) For which values of \( a, b \) is the function
\[
h(x) = \begin{cases} 
\frac{1}{2}(x + 1)^2 & \text{if } x \leq 0 \\
ax + b & \text{if } x > 0
\end{cases}
\]
differentiable on \( \mathbb{R} \)?
Problem 3. (continued)
**Problem 4.** [21 points]

Let $L = 1 - \sqrt{2} < 0$, and let

$$f(x) = \begin{cases} \sqrt{2} - (x - 1)^2 & \text{if } L \leq x \leq 0 \\ \frac{1-x}{(1+x)^2} & \text{if } x > 0 \end{cases}.$$ 

Sketch the graph of $f$ on the interval $(L, +\infty)$ with a detailed labelling. In particular, determine the continuity and differentiability of $f$, identify the intercepts with the coordinate axes, the vertical and horizontal asymptotes, the local and global extrema, the intervals of increase and decrease, and the intervals of convexity and concavity.
Problem 4. (continued)
Additional working space.

If you use this space, please carefully indicate to which problem your work corresponds.