Topological aspects of physics: the quantum Hall effect.

- Classical Hall effect: phenomenon whereby 2-D current carrying conductor (will current $I$) exposed to a transverse magnetic field $B$ develop a potential difference $V_{HH}$ that is transverse to both $I, B$, and conversely.

\[
\text{Define the Hall resistance } \quad R_H = \frac{V_{HH}}{|I|}
\]

Classically, the electrons moving at velocity $v$ are subject to the Lorentz force

\[
\mathbf{F}_L = q \frac{v}{c} \times \mathbf{B}.
\]

Charges accumulate on the boundary of the sample and create an electric field $E$

\[
\text{Friction: } \quad \mathbf{F}_f = -\frac{q}{\mu} V_{HH}
\]

where $\mu$ is the mobility.
Electric Force: \( F = qE \)

At equilibrium:

\[ E + \frac{V}{c} N B - \mu^{-1} N = 0 \]

or in matrix form:

\[
\begin{pmatrix}
E_n \\
E_i
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\mu} & -\frac{B}{c} \\
\frac{B}{c} & 1
\end{pmatrix}
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
\]

We define the resistivity matrix \( \gamma \) by the linear response relation \( E = \gamma j \), where \( j \) is the current density, \( j = q N N \) and \( n \) is the \# of charges per unit area. The conductivity matrix is

\[ \sigma = \gamma^{-1} = \frac{q_n}{\mu^{-2} + \frac{B^2}{c^2}} \begin{pmatrix}
\mu^{-1} & B/c \\
B/c & \mu^{-1}
\end{pmatrix} \]

\( \gamma_H \): Hall conductivity

Remark: in the trochoidal limit: \( j \to 0 \):

If \( B = 0 \):

\[ \lim_{\mu \to 0} \sigma_D = \lim_{\mu \to 0} qn \mu = \infty. \]

unlimited acceleration of the electron in the direction of \( E \).

If \( B \neq 0 \):

\[ \lim_{\mu \to 0} \sigma_D = 0 \text{, no direct current.} \]
but \( \lim_{m \to -\infty} \sigma_H = \frac{qnc}{B} \).

In particular, the Hall conductivity increases with the electron density, namely with the thickness of the sample.

Its inverse is proportional to \( B \).

Experimental result (see Klitzing et al., 1980)

\[ \frac{\hbar c}{qB} \]

\[ \sigma_H \]

quantization!

not reflected in the classical theory at all.

classical value.

here: \( \frac{\hbar}{c} \) are in units of \( q \)

\( \frac{\hbar c}{qB} \) is called the \textit{Klitzing factor} \( \nu \).

Very accurate quantization independent of the details of the experimental setup.

Why!
Model

1. In the continuum, the Landau Hamiltonian:
   \[ H = L^2 (Nz; C) \]  : one-particle Hilbert space

Free electron interacting only with \( B = B_\mathbf{e}_z \) magnetic field

\[ H = \frac{1}{2} \left( \mathbf{p} - \frac{\mathbf{q}}{c} \mathbf{A} \right)^2 \]

where \( \mathbf{A} \) is a multiplication operator given by

\[ \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{X} = \left( \begin{array}{c} -\frac{1}{2} B X_2 \\ \frac{1}{2} B X_1 \end{array} \right) \]

(\( \mathbf{B} \) is magnetic field associated to \( \mathbf{A} \) vector potential \( A \) is its curl: \( \text{curl} \ A = B e_z \) indeed)

Claim 1: \( \text{Spec } H = \{ (2n + 1) \frac{q B}{c} : n \in \mathbb{N} \} \)

and each eigenvalue is infinitely degenerate

Proof: See exercise. (called the "Landau levels")

Claim 2: Each Landau level has \( \frac{q B}{2 \pi c} \) eigenstates per unit area

Proof: See exercise.

Now, the ground state of the (non-interacting) \( N \)-electron system is obtained by "filling" the levels

(by the Pauli Principle)
By Claim 2, we know how many can be put per unit area in each level

The filling factor $\nu = \frac{n}{qB}$ (where $n$ is the electron density, see p. 76) corresponds to the number of occupied Landau levels.

Note: in general, the top Landau level is partially filled.

and if $\nu \not\in \mathbb{N}$, then $\nu + 1 \not\in \mathbb{N}$ ...

(ii) On the lattice: the Hofstadter (or Harper) model

One-particle Hamiltonian $h = e^i(\mathbb{Z}^2)$ will create / annihilate operator $\Delta_{xy}$.

$$H(y, \mu) = \sum_{(x, y) \in \mathbb{Z}^2} e^{i\phi(x, y)} \Delta_{x, y}^z \Delta_{x, y}^w + \mu \sum_{(x, y) \in \mathbb{Z}^2} \Delta_{x, y}^z \Delta_{x, y}^w$$

where

$$e^{i\phi(x, y)} = \int_{\mathbb{R}} e^{imx\phi} \delta_{x-x', y-y'}$$

$(\text{horizontal hopping})$

$(\text{vertical hopping})$

$(\text{only nearest neighbors})$

Note: $\phi$ is called magnetic flux per unit cell.
Under the action of a translation by \( \chi \)
\[ a_{x,y} \rightarrow a_{x+n, y+m} = U_{nm} \, a_{x,y} \, U_{nm}^{-1} \]
we have that
\[ U_{nm} \, \mathcal{H}(\chi, \mu) \, U_{nm}^{-1} = \mathcal{H}(\chi_{nm}, \mu) \]
where
\[ \chi_{nm}(x,y;x',y') = \chi(x-n, y-m; x'-n, y'-m) \]

Under the action of a gauge transformation
\[ a_{x,y} \rightarrow e^{iX(x,y)} \, a_{x,y} = V(x)^{-1} \, a_{x,y} \, V(x) \]
\[ a_{x,y} \rightarrow e^{-iX(x,y)} \, a_{x,y} = V(x)^{-1} \, a_{x,y} \, V(x) \]
we have that
\[ V(x)^{-1} \, \mathcal{H}(\chi, \mu) \, V(x) = \mathcal{H}(\chi + \Delta \chi, \mu) \]
where
\[ \Delta \chi(x,y;x',y') = \chi(x,y;x',y') + X(x,y) - X(x',y') \]

Conclusion: While \( \mathcal{H}(\chi, \mu) \) is invariant under neither \( U_{nm} \) nor \( V(x) \), it is invariant under the so-called magnetic translation:
\[ \tilde{U}_{nm} = e^{iX_{nm}} \, U_{nm} \]
where
\[ \Delta X_{nm} = \chi - \chi_{nm} \]
Note. In general, $\mathcal{L}^2(\mathbb{R}^d,\mathcal{A}) \to \tilde{U}_{\nu,\mu}$ is not a unitary representation:

$$\tilde{U}_{\nu,\mu} \tilde{U}^*_{\nu',\mu'} = e^{iX_{\nu,\mu} U_{\nu,\mu} + iX'_{\nu',\mu'} U^*_{\nu',\mu'}} e^{i(X_{\nu,\mu} + X'_{\nu',\mu'} - (\nu,\mu) - \nu'\mu')},
$$

However, $\psi$ is $\frac{p}{q} \in \mathbb{Q}$ (p,q coprime),

$\mathcal{L}^1(\mathbb{R}^d) \to \tilde{U}_{\chi_{\nu,\mu}}$ forms a representation.

In this case, the one-particle Hamiltonian

$$e^{i\chi(x_0;x',y)}$$

acts on $L^2(\mathbb{R}^d)$ can be diagonalized by Fourier methods. The spectrum (as a subset of $\mathbb{R}$) is $\mathbb{Z}$, see Figure.

Quantization of conductance $\sigma_H$ can be understood (and was historically first understood) in three periods term. We will look at this from a different point of view.
Quantization of conductance: the geometric picture
(also: the periodic case).

(Platonic) setting:

1. A family of Hamiltonians
   \[ H_\Phi, \quad \Phi = (\phi_1, \phi_2) \in \mathbb{T}^2 \]

2. A family of ground state projectors
   \[ P_\Phi, \quad \Phi \in \mathbb{T}^2 \]

3. A spectral gap above the ground state energy

4. Linear response coefficient, the conductance:

   \[ \sigma(\Phi) = -i \text{Tr} \left( P_\Phi \left[ \partial_1 P_\Phi, \partial_2 P_\Phi \right] \right) \]

Claim: Consider

\[ \sigma := \frac{1}{2\pi} \int_{\mathbb{T}^2} \sigma(\Phi) \, d\Phi \]  \hspace{1cm} (1)

Then:

(i) \( \sigma \) is the first Chern number of a certain bundle of \( \mathbb{T}^2 \).

(ii) \( \sigma \in \mathbb{Z} \).

See Arronson, Soler, PRL 54, 1985
Thouless - Kohmoto - Nightingale - den Nijs, PRL 49, 1982

Remarks: The argument requires the additional integration.
over the form, but it applies to interchanging electrons.

In the non-interacting, periodic case, \( (\sigma) \) arises naturally as the linear response coefficient, where however \((\phi, \phi)\) is replaced by momenta \((\mathbf{h}_n, \mathbf{h}_n)\) and the form is the Brillouin zone.

We start with a few words about vector bundles and their invariants.

A differentiable fibre bundle \((E, p, M, F, G)\)

- \(E\) is a differentiable \(\text{(total space)}\)
- \(M\) \(\text{(base space)}\)
- \(F\) \(\text{(fibre)}\)
- \(p : E \rightarrow M\) is a surjection.

\[ p^{-1}(x) = F_p \cong F \text{ is the fibre at } x. \]

- \(G\) is a (Lie) group acting on \(F\)
- Atlas \(\{U_i\}\) of \(M\) will beomorphisms

\[ \phi_i : U_i \times F \rightarrow p^{-1}(U_i) \quad \text{"local trivialization"} \]

such that \( (p \circ \phi_i)(x, f) = x \)

- If \(U_i \cap U_j \neq \emptyset\), the transition function

\[ t_{ij}(x) = \phi_i^{-1} \circ \phi_j \quad (\phi_i^{-1}(x) = \phi_j(x)) \]
is an element of $G$.

In other words:

$$\phi_j(x, 1) = \phi_i(x, t_{ij}(x)y)$$

Notation: $E \xrightarrow{p} M$

A bundle $E \xrightarrow{p} M$ is 
\textit{trivial} if $t_{ij}$ can be taken to be the identity for all $i, j$. Then

$$E = M \times F$$

globally.

A section $s: M \rightarrow E$ is a differentiable map st.

\((\text{global})\) $p \circ s = \text{id}_M$.

If $s$ is defined only on a neighborhood $U_i$, we call it a \underline{local section}. There may be no global section.
A vector bundle is a fibre bundle where if is a vector space, \( M \) dimension \( m \), the group is \( G = GL(m; \mathbb{C}) \).

Example: Tangent bundle \( TM \) of an \( m \)-dimensional real manifold \( M \), where \( \mathcal{T} = \mathbb{R}^m \).

Sections of \( TM \) are vector fields.

A line bundle is a vector bundle with \( \dim F = 1 \).

Example: A cylinder \( S_1 \times \mathbb{R} \to S_1 \) is a trivial \( \mathbb{R} \)-line bundle.

Another example:

(i) \( \mathcal{H} \) a Hilbert space, \( M \) a manifold and \( E \subset \mathcal{H} \) a subspace of \( \mathcal{H} \) which depend smoothly on \( x \in M \). Define

\[
E := \{ (\nu, x) \in \mathcal{H} \times M : \nu \in E_x \} \subset \mathcal{H} \times M.
\]

and \( p : E \to M \) is defined by

\[
(\nu, x) \mapsto x.
\]

Notes: Even though \( \mathcal{H} \times M \) is a trivial bundle and \( E \) is a subbundle of it, \( E \) may be non-trivial.

For example: \( E_x \) is the eigenspace of \( \mathbb{A} \) for an isolated isolated eigenvalue \( \lambda(x) \).
(ii) \( \mathcal{U} = S^1 \text{ and } F = \mathbb{C}^1, \mathbb{C}^2 \).
\( U_1 = \mathbb{C} \backslash [-1, 1] \text{ and } U_2 = \mathbb{C} \backslash [1, 1]. \)
\( U_1 \cap U_2 = \mathbb{C} \backslash \{0\}. \)

We specify the transition functions \( \tau_i(x) : F \to F \) for all \( x \in U_1 \cap U_2 \).

Possibility 1: \( \tau_{12}(x)(v) = v \) \( x \in \mathbb{C} \backslash \{0\}, \forall v \in F. \)

Possibility 2: \( \tau_{12}(x)(v) = \overline{v} \) \( \Im(x) > 0, \forall v \in F. \)

\( \mathbb{C} \)

Cylinder: \( G = \{ e^i \} \)

\( \mathbb{C} \)

Kähler strip.
\( G = \{ e^i, i \} \approx \mathbb{Z}_2. \)
where \( i : v \mapsto -v. \)

Proposition: A vector bundle \( E \to M \) will be \( \text{hom}(F) = n < \infty \) is trivial if there are \( n \) global sections \( \{ s_i : 1 \leq i \leq n \} \) such that

\[ \text{span}\{ s_i(x) : 1 \leq i \leq n \} = F_x \]

for all \( x \in M. \)

(\text{v.e.}, \( s(x) \in \mathbb{C}^1(x) = F_x \) for any section.)
Proof: Assume that such sections exist. Let \( u \in E, \ p(u) \in M \) and \( U = p(u) \). The second component of \( \Phi''(u) \) is an element of \( F_p(u) \) and it can therefore be expanded in the basis \( \{ s_n(p(u)), \ldots, s_N(p(u)) \} \). Gathering the coefficients in a vector of \( C^n \), \( (u_1, \ldots, u_n) \), we define a map

\[ \Psi : E \to M \times C^n \]

\[ u \mapsto \Psi(u) = (p(u), u_1, \ldots, u_n) \]

Check: \( \Psi \) is an isomorphism of fibre bundles.

Reciprocally, if \( E \) is trivial, there is

\[ \Phi : E \to M \times C^n \]

and we define \( s_i(x) = \Phi^{-1}(x, e_i) \), where \( \{ e_1, \ldots, e_n \} \) is the canonical basis of \( C^n \).

We turn to connections and curvature of vector bundles, keeping in mind the example on page 84:

\( E \) is a subbundle of \( \mathcal{H} \times M \to M \) given by fibres \( \mathcal{H}_\phi = C^n \), \( \phi \in M \).

We think of \( \mathcal{H}_\phi \) as being determined by a smooth family of orthogonal projections \( \Pi_\phi : \mathcal{H} \to \mathcal{H}_\phi \).
\[ \mathcal{H}_\phi = P_\phi \mathcal{H}. \]

- Recall, a section \( \sigma : E \rightrightarrows \mathcal{M} \) is a map \( p \circ \sigma = \text{id}_\mathcal{M} \).

and denote \( \Gamma(E) \) the space of sections.

In this example, \( \sigma \) is a smooth choice of vectors in the range of \( P_\phi : \sigma(\phi) \in \mathcal{H}_\phi \).

- A connection on \( E \) is a map \( \nabla : \Gamma(E) \to \Gamma(E \otimes \mathcal{T}_\mathcal{M}^*) \)

that satisfies a Leibniz rule:

\[ \nabla(\sigma \phi) = \nabla(\sigma) \phi + \sigma \delta \phi. \]

for any \( \phi \in C^\infty(\mathcal{M}) \).

In the example: let \( \phi \in \mathcal{M} \). For any vector \( X \in T_\mathcal{M} \mathcal{M} \),

\[ \nabla_X(\sigma) \]

is a vector in the range of \( P_\phi \).

(\( \nabla \) is an assignment of the directional derivative of \( \sigma(\phi) \) along curves given by vector fields in \( T_\mathcal{M} \mathcal{M} \)).

- Let \( X \in \mathcal{T}_\mathcal{M} \) be a vector field on \( \mathcal{M} \). A connection induces a covariant derivative.
\[ \nabla_X : \Gamma(E) \rightarrow \Gamma(E) \]
\[ s \rightarrow \nabla_X(s) = \nabla(s)(X) \]

Note: \( \nabla \) is a linear on \( E \):
- \( \nabla_X(\ell s_1 + s_2) = \nabla_X(s_1) + \nabla_X(s_2) \)
- \( \nabla_X \) is linear on \( X \):
  - \( \nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s) \)
  - \( \nabla_X(\psi s) = \psi \nabla_X(s) + X(\psi)s \)
  - \( \nabla_{\phi X}(s) = \phi \nabla_X(s) \)

Now, the curvature associated to a connection \( \nabla \) is a map:
\[ R : \Gamma(E) \rightarrow \Gamma(E \otimes \Lambda^2 \mathfrak{X}(E)) \]

with:
\[ R(X,Y)(s) = \nabla_X \nabla_Y(s) - \nabla_Y \nabla_X(s) - \nabla_{[X,Y]}(s) \]

Example: The Berry connection is given by
\[ (\nabla_X \Phi)(\phi) = P_{\phi} X(\phi \cdot \phi) \]

input a function \( \phi : E \rightarrow C \),

is a vector in
\[ T_{\Phi(\phi)}E \subset E_{\Phi} = P_{\Phi} \mathfrak{X} \]

the tangent space at any point is isomorphic to the vector space itself.
Note that in this case

\[ \nabla - P d \]

where \( d \) is the exterior derivative on \( \mathcal{P} \). If \( \phi \in \Gamma(E) \),

\[ ds = d(\phi s) = (dp)s + Pds \]

namely

\[ \nabla = d - (dp)P \]

see the informal discussion of \( \parallel \)-transport in the advective section of these notes.

Finally, given a connection \( \nabla \), one defines a parallel transport: An assignment to every curve \( \gamma: [0, 1] \rightarrow \mathcal{M} \) of \( \gamma \) a map

\[ T_{t,s}: E_{\gamma(t)} \rightarrow E_{\gamma(s)} \]

s.t.

(i) \( T_{s,s} = \text{id}_{E_{\gamma(s)}} \)

(ii) \( T_{t,s} \circ T_{s,r} = T_{t,r} \)

(iii) \( \nabla_{\gamma'} (T_{t,s}(u)) = 0 \)

In the example: \( \mathcal{A}_t = \mathcal{A}_{t,0} \) is parallel transported along \( \gamma \)

\[ 0 = \nabla_{\dot{\gamma}} \mathcal{A} = \dot{\mathcal{A}} - P \mathcal{A} \]
which is equivalent to \( \mathbf{P} \mathbf{c} = 0 \), our
previous definition of parallel transport.

A calculation yields: the Riemann curvature is given by

\[
R(X, Y) = [X^P, Y^P]^P
\]

In the example, and with \( X = \partial_t = \frac{\partial}{\partial t} \), \( Y = \partial_c = \frac{\partial}{\partial c} \),

\[
R(\partial_t, \partial_c) = (P \partial_t P \partial_c - P \partial_c P \partial_t)^P \quad ([\partial_t, \partial_c] = 0)
\]

\[
= P(\partial_t P \partial_c - \partial_c P \partial_t)^P = P((\partial_t P)(\partial_c P) - (\partial_c P)(\partial_t P))^P
\]

Since \( P(\partial_t P)P(\partial_c P) = 0 \) since \( \partial_t \mathbf{r} \) is off-diagonal.

If \( \text{dim}(\mathcal{M}) = 2 \) and since \( \mathbf{R} \) is a two-form,
we can integrate it over \( \mathcal{M} \). In particular,

\[
\text{Tr}(\mathbf{R}) \text{ is a } \mathbb{R} \text{-valued two-form on } \mathcal{M} \text{ and we define }
\]

\[
\text{Ch}_n(E) = \frac{1}{2\pi i} \int_{\mathcal{M}} \text{Tr}(\mathbf{R})
\]

\[
= \frac{1}{2\pi i} \int_{\mathcal{M}} \text{Tr}(P(\partial_t P)(\partial_c P))d\Phi_t d\Phi_c.
\]

The first Chern number of the bundle.
We now turn to the actual computation of the Chern number, in the case \( \text{Rash}(P_\Phi) = 1 \), namely that of a line bundle over \( T^2 \).

**Theorem:** \( \sigma \in \mathbb{Z} \)

Note that \( \overline{\sigma(\Phi)} = i \text{Tr} \left( [\partial, \Phi] P \right) = \sigma(\Phi) \) by cyclicity, so that \( \overline{\sigma(\Phi)} \in \mathbb{Z} \) \( \forall \Phi \in \pi_1 \).

**Proof:** Let \( U \) be a neighborhood of \( T^2 \) and let \( \psi_\Phi \) be a local section \( \psi : U \to E \), such that \( \|\psi_\Phi\| = 1 \) \( \forall \Phi \in U \). In particular

\[
P_\Phi = \langle \psi_\Phi, \cdot \rangle \psi_\Phi.
\]

Since \( \text{Rash} P_\Phi = 1 \).

Let \( \nu \in T U \) be a vector field

\[
\nu_\Phi = \begin{pmatrix}
\langle \psi_\Phi, \partial_1 \psi_\Phi \rangle \\
\langle \psi_\Phi, \partial_2 \psi_\Phi \rangle
\end{pmatrix}
\]

We claim that

\[
\text{Tr} \left( P_\Phi \left[ \partial_1 P_\Phi, \partial_2 P_\Phi \right] \right) = \text{curl} \, \nu_\Phi
\]

Indeed,

\[
\partial_2 P_\Phi = \langle \partial_2 \psi_\Phi, \cdot \rangle \psi_\Phi + \langle \psi_\Phi, \cdot \rangle \partial_2 \psi_\Phi.
\]

Thus \( \langle \psi_\Phi, \partial_2 \psi_\Phi \rangle = 1 \) implies \( \langle \partial_2 \psi_\Phi, \psi_\Phi \rangle = -\langle \psi_\Phi, \partial_2 \psi_\Phi \rangle \).

So that

\[
\partial_2 P_\Phi (\psi_\Phi) = \langle \partial_2 \psi_\Phi, \psi_\Phi \rangle \psi_\Phi + \partial_2 \psi_\Phi.
\]
and furthermore

\[ \left\langle \Psi_{\Phi}, \cdot \right\rangle \partial_i \Phi = \left\langle \partial_i \Psi_{\Phi}, \cdot \right\rangle + \left\langle \Psi_{\Phi}, \partial_i \Psi_{\Phi} \right\rangle \left\langle \Phi, \cdot \right\rangle. \]

Now (summation over \(ij\) implied)

\[ \sigma(\Phi) = \varepsilon_{2ij} \text{Tr} \left( \Psi_{\Phi} \left\langle \Psi_{\Phi}, \cdot \right\rangle \partial_i \Phi \partial_j \Phi \right) \]

\[ = \varepsilon_{2ij} \left( \left\langle \partial_i \Psi_{\Phi}, \cdot \right\rangle + \left\langle \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle \left\langle \Phi, \cdot \right\rangle \right) \]

\[ \left( \left\langle \partial_j \Psi_{\Phi}, \cdot \right\rangle + \frac{2}{\varepsilon} \left\langle \Psi_{\Phi}, \theta \Psi_{\Phi} \right\rangle \right) \]

\[ = \varepsilon_{2ij} \left( \left\langle \partial_i \Psi_{\Phi}, \Psi_{\Phi} \right\rangle \left\langle \partial_j \Psi_{\Phi}, \Psi_{\Phi} \right\rangle + \left\langle \partial_i \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle \right) \]

\[ \left( \left\langle \Psi_{\Phi}, \partial_i \Psi_{\Phi} \right\rangle \left\langle \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle \right) \]

and the first term vanishes by antisymmetry of \(\varepsilon\).

Moreover:

\[ \left\langle \partial_i \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle = \partial_i \left\langle \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle - \left\langle \partial_i \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle \]

and the second term vanishes with \(\theta\) again. Hence

\[ \sigma(\Phi) = \partial_i \left\langle \Psi_{\Phi}, \partial_j \Psi_{\Phi} \right\rangle - \partial_j \left\langle \Psi_{\Phi}, \partial_i \Psi_{\Phi} \right\rangle = \nabla \cdot \nabla \Phi \]

This of course holds only on the chart \(U\).

**Fact.** If \(U\) is contractible, then all vector bundles over \(U\) are isomorphic. Hence they are all trivial: \(U \times \mathbb{C}^n\).
We now make two cuts on the torus to obtain a square \( \mathbb{T} \times [-\pi, \pi] \subseteq \mathbb{R}^2 \)
and there exists a global section on \( E = \mathbb{T} \times \mathbb{C} \),
\( \Phi_L, \Phi_R \) such that \( \| \Phi_L \| = 1 \quad \forall \Phi \in \mathbb{T} \).

Define two continuous functions \( \Theta_L, \Theta_R : [-\pi, \pi] \to \mathbb{R} \) by
\[
\Phi_L(\phi, \pi) = e^{i \Theta_L(\phi)} \Phi_L(\phi, -\pi), \quad \Phi_R(\pi, \phi) = e^{i \Theta_R(\phi)} \Phi_R(-\pi, \phi).
\]
(They exist by the normalization condition)

Now, there are two different ways to reach \( \Phi_L(\pi, \pi) \):
\[
e^{i \Theta_L(\pi)} \Phi_L(-\pi, -\pi) = e^{i \Theta_L(\pi)} \Phi_L(-\pi, \pi) = e^{i \Theta_L(\pi)} \Phi_L(-\pi, \pi).
\]

Hence, \( \Theta_L(\pi) + \Theta_L(-\pi) - \Theta_R(\pi) - \Theta_R(-\pi) \in 2\pi \mathbb{Z}. \ (\bigstar) \)

It remains to compute the curl:
\[
\langle \psi_L(\phi, \pi), \partial_1 \psi_L(\phi, \pi) \rangle = \langle \psi_L(\phi, \pi), e^{i \Theta_L(\phi)} (i \Theta'_L(\phi) \psi_L(\phi, \pi) + \partial_1 \psi_L(\phi, \pi)) \rangle
\]
\[
= i \Theta'_L(\phi) + \langle \psi_L(\phi, -\pi), \partial_1 \psi_L(\phi, -\pi) \rangle.
\]
and the same holds for the second component.

Hence:

\[
\text{Ch}_1(E) = \frac{1}{4\pi i} \int_{\Gamma_1} \text{curl} \, \nu_{\phi} \cdot d\Phi - \frac{1}{4\pi i} \int_{\Gamma_2} \text{curl} \, \nu_{\phi} \cdot d\Phi
\]

Sideways:

\[
\frac{1}{2\pi i} \int_{\Gamma_1} \nu_{\phi} \cdot d\Phi.
\]

Gathering contributions of opposite sides of the square:

\[
\text{Ch}_1(E) = \frac{1}{2\pi i} \int_{\partial \Phi_1} (\nu_{(\Phi_1, \Phi_2)}^1 - \nu_{(\Phi_1, \Phi_2)}^2) d\Phi_2
\]

\[+ \frac{1}{2\pi i} \int_{\partial \Phi_2} (\nu_{(\Phi_2, \Phi_1)}^2 - \nu_{(\Phi_2, \Phi_1)}^1) d\Phi_1.
\]

\[= \frac{1}{2\pi i} \int_{\Phi_1} (i \Theta'_1(\phi) + i \Theta'_2(\phi)) d\phi.
\]

\[= \frac{1}{2\pi i} \left(-\Theta_1(\pi) + \Theta_1(-\pi) + \Theta_2(\pi) - \Theta_2(-\pi)\right) \in \mathbb{Z}
\]

by (\text{(ii)})

From the proof, we see that if there were a global section, then \(\Theta_1 = \Theta_2 = 0\) and hence \(\text{Ch}_1(E) = 0\).

Reciprocally:

Proposition. If \(\text{Ch}_1(E) = 0\), then there is a global section on \(E\), and \(\text{ch} \| \mathcal{F} \| = 1\).
Proof. First, we claim that we can always construct a global section on the cylinder
\[ \mathbb{T}^2 = [-\pi, \pi] \times S^1. \]
Indeed, given the initial section \( \tilde{\psi}_\phi \) on \( \mathbb{T}^2 \), we define another one
\[ \tilde{\psi}_\phi = e^{i\alpha(\phi)} \psi_\phi \]
where \( \alpha: \mathbb{T}^2 \to \mathbb{R} \) is a continuous function. Its corresponding Hamilton function is fixed by
\[ \tilde{\psi}_\phi(\phi, \pi) = e^{i\alpha(\phi, \pi)} \psi_\phi(\phi, \pi) = e^{i\alpha(\phi, \pi)} \psi_\phi(\phi, \pi). \]
Hence \( \tilde{\Theta}_1(\phi) = 2\pi n \), constant \( \forall \phi \in \mathbb{T} \)
\[ \alpha(\phi, \pi) + \tilde{\Theta}_1(\phi) - \alpha(\phi, -\pi) \in 2\pi \mathbb{Z}. \]
so we can pick, for any \( n \in \mathbb{Z} \),
\[ \alpha(\phi, \phi) = \frac{\tilde{\Theta}_1(\phi) - \tilde{\Theta}_2(\phi)}{2\pi} \quad \text{(or any continuous function s.t.} \quad \tilde{\Theta}_2(\pi) - \tilde{\Theta}_2(-\pi) = \pi). \]
Hence \((\#)\) expressed will \( \tilde{\Theta}_1 \) give
\[ \text{Ch}_1(E) = \frac{1}{2\pi} \left( \tilde{\Theta}_2(\pi) - \tilde{\Theta}_2(-\pi) \right) \]
since \( \tilde{\Theta} \) is constant. Hence \( \text{Ch}_1(E) \) is the winding number of the map
\[ S^1 \ni \phi \mapsto e^{i\tilde{\Theta}(\phi)} \in U(1) \]
Assume now that $\alpha$ can further be chosen so that $\bar{\Theta}_2$ is constant and $\in \text{B}_2 \mathbb{Z}$. Then
\[
\alpha(\Phi, \phi_i) = \frac{\bar{\Theta}_2(\phi_i) - \Theta_2(\phi_i)}{2\pi} \phi_i
\]

But $\alpha(\Phi, \tau) - \alpha(\Phi, \tau_1) \not\in 2\pi \mathbb{Z}$ since otherwise $\bar{\Theta}_1$ would be lost. Hence
\[
\Theta_2(\tau) - \Theta_2(\tau_1) \not\in 2\pi \mathbb{Z} \quad \text{for all} \quad \Phi_i
\]
\[
\Phi_i = -\text{Ch}_i(E) \phi_i \in 2\pi \mathbb{Z}
\]
\[
\Phi_i = \text{Ch}_i(E) = 0.
\]

In other words: If $\text{Ch}_i(E) = 0$, (46) holds and hence
\[
e^{i\bar{\Theta}_1(\Phi)} = e^{i\bar{\Theta}_2(\Phi)} = 1 \quad \text{for all} \quad \Phi_i, \phi_i \in \text{B}_2 \mathbb{Z},
\]
so that $e^{i\alpha(\Phi)} \Phi$ is a global section.

Finally, for a general fibre bundle of rank $N$:
(i) There exist $N$, linearly independent, global sections of $\bar{\pi}^2$.
(ii) Let $\Psi_i, 1 \leq i \leq N$ be these sections. Let
\[
\Psi_i^\dagger(\tau, \phi_i) = \sum_{l=1}^N T(\phi_i)_{ij} \Psi_j^\dagger(-\tau, \phi_i)
\] where $T_2 : S^1 \rightarrow U(N)$.
(iii) $\text{Ch}_i(E)$ is the winding number of $\phi_i$ under $T_2(\phi)$. (47)
In particular, \( \text{Ch}_n(\mathcal{E}) \in \mathbb{Z} \) in this case, too.

We have proved quantization of the Hall conductance in a very general setting, but under the assumption that \( \Omega \) is defined through an integration over the torus. It remains to prove that

\[
\Phi \mapsto \sigma(\Phi) = \text{Tr}(P_\Phi [\Omega, P_\Phi, \partial_\Phi])
\]

is constant over \( \mathbb{T}^2 \). Note that

\[
\sigma(\Phi) = \text{Tr}(P_\Phi [\Omega, H_\Phi, \partial_\Phi(\partial_\Phi)])
\]

where \( \Omega(\partial_\Phi) \) are the local generator of parallel transport. It is this representation that allows for the use of the Lieb-Robinson bound that lead to a recent proof of constant curvature, see Hastings-Tikhomirov CTR 234 2015 Bednorz-Ho-DeRoeck-Franz AHF 19 2018.
Constant curvature in the many-body setting.

(1) The physical setting: square lattice on a torus
\(([0, L] \times [0, L]) \cap \mathbb{Z}^2 = \Lambda\)

A Hamiltonian:
\[ H_\Lambda = \sum_{x \in \Lambda} \Phi(x) \]

with finite range: \(\Phi(2) = 0, \forall \text{ dist}(x) > R\).

Charge operator: at each site \(Q_{\mathbf{x} \mathbf{y}} = Q_{\mathbf{x}}\) will
\[ \text{Spec}(Q_{\mathbf{x} \mathbf{y}}) \subset \mathbb{Z} \]
e.g. \(Q_{\mathbf{x} \mathbf{y}} = b^*_\mathbf{x} b_\mathbf{y}\) for lattice fermion.

Moreover \(Q_{\mathbf{x}} = \sum_{x \in \Lambda} Q_{\mathbf{x} \mathbf{y}}\)

Assumption 1: local charge conservation: \(X \in \mathbb{Z} \cap \Lambda\) implies
\[ [Q_{\mathbf{x}}, \Phi(x)] = 0 \]

usually, local gauge invariance: \(\Phi(x) = e^{i\phi_2} \Phi(x) e^{-i\phi_2}\)

Special set:
\[ \mathbb{G} \]

\(X_{\mathbf{y}}, \text{charge } Q_{\mathbf{y}}\)

\(\mathbb{G}\)

\(X_{\mathbf{y}}, \text{charge } Q_{\mathbf{y}}\)

\(\mathbb{G}\)
By charge conservation, the current operators

\[ i \left[ H, Q_j \right] = i \sum_{\text{tca}} \left[ \langle \Phi(\tau) \rangle, Q_j \right] \]

are supported on two disjoint (i.e. \( L > 2R \)) ribbons:

\[ \forall \phi \left( i \left[ H, Q_j \right] \right) \in \mathcal{X}_j^- \cup \mathcal{X}_j^+ \]

**Note:** If a current actually flows, the expectation value of the current operator will be zero by charge conservation and we need to isolate one of the ribbons and "measure" current through \( \mathcal{X}_j^- \).

The same holds for the driving (e.g. a tension): if the potential increases at \( 0 \), then it must decrease at \( 1/T \).

Here this can be done by making "gauge transformation" on some of the interaction terms only:

Let \( U_j(\phi) = e^{-if\phi Q_j} \), which are periodic. Then

\[ H(\Phi_-, \Phi_+) = \sum_{\text{tca}} U_1(\Phi_2)^T U_1(\Phi_2)^T \Phi(\tau) U_1(\Phi_+^T) U_1(\Phi_2) \]

\[ \Phi_+ = \left( \begin{array}{c} \Phi_2^- \end{array} \right), \quad \Phi_2^+ = \left( \begin{array}{cc} \Phi_2^- & 1 \end{array} \right) \]

\[ \Phi_2^- = (\Phi_-, \Phi_+^T) \]

\[ U_1(\Phi_2)^T \Phi(\tau) U_1(\Phi_2) \]

\[ \begin{cases} Q_j^+ \mid \text{in } \mathcal{X}_j^- + \phi \\ Q_j^- \mid \text{in } \mathcal{X}_j^+ \\ 0 \mid \text{otherwise} \end{cases} \]
In picture:

\[ \Phi_i \rightarrow \Phi_i^t \rightarrow \Phi_i^{-t} \rightarrow \Phi_i \]

And this is a unitary transformation of the Hamiltonian if and only if
\[ \Phi_i^{-t} = -\Phi_i^t ; \Phi_i = \Phi_i^t \cdot \]

In other words: For any \( \Theta \in T^2 \):

\[ H(\Phi_+ + \Theta, \Phi_+ - \Theta) = U_2(\Theta)^* U_1(\Theta)^* H(\Phi_-, \Phi_+) U_1(\Theta) U_2(\Theta) \]

Define: "Twist Hamiltonian" \( \tilde{H}(\Phi) = H(\Phi, 0) \)

"Twist-dual Hamiltonian" \( H(\Phi) = H(\Phi_+ - \Phi) \)

Importantly: \( H(\Phi) \) are a unitarily equivalent family, \( \tilde{H}(\Phi) \) are not.

Moreover:
\[ \text{supp} (\tilde{H}(\Phi) - H) \subset \mathbb{R} \chi_i \cup \mathbb{R} \chi_i^t \]
\[ \text{supp} (\tilde{H}(\Phi) - H(\Phi)) \subset \mathbb{R} \chi_i \cup \mathbb{R} \chi_i^t \]

**Assumption 2:** \( \tilde{H}(\Phi) \) are all gapped, uniformly in \( \Phi \in T^2 \)

and \( L \geq 2N \), with ground state projection \( \tilde{\rho}(\Phi) \) such that \( \text{Tr}(\tilde{\rho}(\Phi)) = 1 \)

Linear response coefficient:
\[ \sigma(\Phi) = -i T_i (\tilde{\rho}(\Phi) [2_1 \tilde{\rho}(\Phi), 2_2 \tilde{\rho}(\Phi)]) \]

(i) The mathematical statement:
Proposition: \[ \sup_{\phi, \phi' \in \mathcal{F}} |\sigma(\phi) - \sigma(\phi')| \leq C_N L^{-N} \]
for all \( L > L_0 \) and \( N \in \mathbb{N} \), where \( C_N \) is independent of \( L \).

Consider curvature for large \( L \).!

\[ \sigma_j \tilde{\Psi} = i [\tilde{\kappa}_j, \tilde{\Psi}] \]
where \( \tilde{\kappa}_j(t) = \tilde{\eta}_j(t\tilde{H}) = \int \omega(t) e^{it\tilde{H}} \tilde{\eta}_j \tilde{\eta}_j e^{-it\tilde{H}} dt \).

Two properties: (i) By the Lieb–Robinson bound and \( \text{supp} (\sigma_j \tilde{\Psi}) \subset 2X_j \), \( \tilde{\kappa}_j \) is approximately supported in \( 2X_j \).

(ii) Since \( \sigma_j \tilde{\Psi} \) is a sum of local terms, so is \( \tilde{\kappa}_j \), approximately.

Now, the same holds for the twist–antitwist Hamiltonian \( H(\phi) \) and the corresponding \( \kappa_j(\phi) \), but their support is in \( 2X_j \cup \bar{2X}_j \).

Hence: 1) \( [\tilde{\kappa}_1, \tilde{\kappa}_1] \) is supported in the corner \( \Delta \)
2) \( [\kappa_1, \kappa_1] \) is supported in your corners, and it is a sum of four operators, each supported in just one corner.
\[ [K_1, K_2]_\Delta \approx [\tilde{K}_1, \tilde{K}_2] \]

the restriction of \([K_1, K_2]_\Delta\) to \(\Delta\).

Furthermore, since \(\tilde{H}(\Phi) - H(\tilde{\Phi})\) is supported on \(\mathcal{D}X^+ \cup \mathcal{O}X^+\),

(namely \(L^2 - L_2\) away from \(\Delta\)), their

ground states yield (approximately) equal expectation values for local observables in \(\Delta\).

Altogether,

\[
\tau(\Phi) = -i \text{Tr} \left( \hat{P} \left[ \omega, \hat{P}, \Xi^+(\hat{P}) \right] \right) \\
= i \text{Tr} \left( \hat{P} \left[ \tilde{K}_1, \tilde{K}_2 \right] \right) \\
= i \text{Tr} \left( \tilde{P} \left[ K_1, K_2 \right]_{\Delta} \right) \\
= i \text{Tr} \left( P(\Phi) \left[ K_1(\Phi), K_2(\Phi) \right]_{\Delta} \right)
\]

But the Hamiltonians \(H(\Phi)\) are unitarily equivalent

\[
H(\tilde{\Phi}) = e^{i(\Phi, \Omega + \Phi(\omega))} H(\bar{\Phi}) e^{-i(\Phi, \Omega + \Phi(\omega))}
\]

The same holds for \(P(\Phi)\) and \(K_j(\Phi)\). Since

the unitary is a product of one-site terms, they commute

with the restriction \((\cdot)_{\Delta}\). Cyclicity of the trace yields

the claim.
An analytic approach: Index of projections

We start with a very vague description of Laughlin's argument for the quantization of $\tau$. It should be taken here not as an argument, just as a motivation for what comes next.

Physical setting:

\[ \phi \]

\[ E \]

Increasing the magnetic flux through the system creates an electromotive force $E$ in the angular direction. If a non-zero magnetic field $B$ is present, this induces a Hall current in the longitudinal direction, from right to left.

Hamiltonian: Under Hamiltonian, will steep potentials to model the edges.

Approximate spectral picture: recall that there is a constant density of states in each level.
A computation shows: As $\phi$ increases, the localization centers of the states move to the left; an increase of $\frac{e}{\hbar}$ moves is simply a shift of the picture by one: hence charge transport occurs at the edge and hence

$$\Delta Q \propto (\# \text{ of occupied Landau levels}).$$

This proves quantization since $\sigma \propto \Delta Q$.

What happens? Let $P$ be the infinite dimensional projection onto the occupied Landau levels (aka "Fermi projection"). As $\phi$ changes, it induces a unitary transformation $U$ in the Hilbert space. Hence the charge transport can be expressed as

$$\text{Tr} \left( U P U^\dagger - P \right)$$

Note that: (i) Both $\text{Tr} P$ and $\text{Tr}(U P U^\dagger)$ are ill-defined, but the trace of their difference may be finite.

(ii) If the trace really counts a number of states, then we may as well compute

$$\text{Tr} \left( (U P U^\dagger - P)^{2n+1} \right)$$

which may be better behaved (likewise as a sum of eigenvalues of modulus < 1).