Adiabatic Theorems

Setting: a quantum dynamics that depend smoothly on a parameter $\tau \in \mathcal{M}$, where $\mathcal{M}$ is a smooth manifold.

Typically: $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\tau \rightarrow \mathbf{H}_\tau = \mathbf{H}_0$

will $\parallel \mathbf{H}_\tau \parallel$ uniformly bounded.

Let $\Phi_\sigma \in \mathcal{H}$, $\| \Phi_\sigma \| = 1$ be a family of eigenvectors of $\mathbf{H}_0$: $\mathbf{H}_0 \Phi_\sigma = E_\sigma \Phi_\sigma$

Then of course $e^{-it\mathbf{H}_0} \Phi_\sigma = e^{-itE_\sigma} \Phi_\sigma$

namely: the state $\omega_\sigma = \langle \Phi_\sigma, \cdot, \Phi_\sigma \rangle$ is invariant under time evolution.

The question: what happen if the dynamics change slowly with time: $\mathbf{H}_t = \mathbf{H}_0(t)$?

If the dynamics start at $\Phi(0) = \Phi_\sigma(0)$, do we have that $\Phi(t) \sim \Phi_\sigma(t)$ in some sense?

The answer: mostly yes, via adiabatic theorems.
Consider a smooth curve

\[ s \to H_\sigma(s) \in \mathbb{R} \cap [0, 1] \]

of uniformly bounded Hamiltonians. Sh

1) \( \|H_\sigma\| \), \( \|H_{\sigma'}\| \) are uniformly bounded.

2) For each \( s \in [0, 1] \),

\[ \text{Spec} (H_{\sigma(s)}) = \Sigma^{(1)}_{\sigma(s)} \cup \Sigma^{(1)}_{\sigma(s)} \]

will \( \Sigma^{(1)}_{\sigma(s)} \cap \Sigma^{(1)}_{\sigma(s)} = \emptyset \).

and \( \text{inf} \\text{dist} (\Sigma^{(1)}_{\sigma(s)}, \Sigma^{(1)}_{\sigma(s)}) \geq g > 0 \).

Let \( P_{\sigma(s)} \) be the spectral projector corresponding to \( \Sigma^{(1)}_{\sigma(s)} \).

Describing a "slow" change: rescale physical time \( t \) by a factor \( \varepsilon >> 1 \) and consider

\[ s = \varepsilon t, \quad \frac{d}{dt} = \varepsilon \frac{d}{ds} \]

so that \( s \in [0, 1] \) corresponds to \( t \in [0, \varepsilon^{-1}] \).

Correspondingly:

\[ \Psi_\varepsilon(s) := \Psi(t), \quad \text{solve} \]

\[ i \frac{d}{dt} \Psi(t) = H(t) \Psi(t) \quad \Rightarrow \quad i \varepsilon \frac{d}{ds} \Psi_\varepsilon(s) = H_s \Psi_\varepsilon(s). \]

(4)

\[ \text{For the projector:} \]

\[ \begin{cases} \varepsilon \dot{P}_\varepsilon(s) = -i [H_s, P_\varepsilon(s)] \\ P_\varepsilon(0) = P_0 \end{cases} \quad (5) \]
Theorem (Bouin-Katchal, Uchô). Let $P_\varepsilon(s)$ be the solution of (S) with $P_\varepsilon(0) = P_0$. If (1), (1c) hold, then

$$\| P_\varepsilon(s) - P_0 \| \leq C \varepsilon$$

for all $s \in [0,1]$.

Remarks:
(i) If $\Re_\varepsilon(P_0) = 1$, then $\Re_\varepsilon(P_\varepsilon(s)) = \Re_\varepsilon(P_\varepsilon(1)) = 1$ and so

$$| \langle \dot{\Psi}_\varepsilon(s), \dot{\Psi}_s \rangle | = \| P_\varepsilon(s) \dot{\Psi}_s \| \geq 1 - \| (P_\varepsilon(s) - P_s) \dot{\Psi}_s \| \geq 1 - C \varepsilon$$

(ii) Although we will use the gap, a similar bound holds under more smoothness assumptions on $s \to P_0$ without a gap, with a weaker bound $O(1)$ (Avron, Elgart).

(iii) For a smooth $H_s$, the bound can be improved to $C \varepsilon e^{-\tilde{C} \varepsilon}$ for any $\tilde{C} > 0$ provided

$$H(s)(s) = H(0)(1)$$

for all $s \in \mathbb{R}$, and further

$$b = C e^{-2\pi \varepsilon}$$ (Nenciu).

Before we prove the adiabatic theorem, a few useful

(i) A formula for spectral projection (Riesz):
\[ P = -\frac{1}{\imath \Omega} \int_{\Gamma} (H - \varepsilon)^{-1} \, dz. \]

\[ \psi_s = \frac{1}{\imath \Omega} \int_{\Gamma} (H - \varepsilon)^{-1} H_s (H_s - \varepsilon)^{-1} \, dz \]

and so \( \psi_s \) is a holomorphic function.

For any differentiable family of projection
\[ \dot{P}_s = \dot{P} \Rightarrow P_s \dot{P}_s + \dot{P}_s P_s = \dot{P} \]

which further implies:
\[ * P_s \dot{P}_s P_s = 0 \]
\[ * (H - P_s) \dot{P}_s (H - P_s) = 0 \]
\[ * \dot{P}_s = P_s \dot{P}_s (H - P_s) + (H - P_s) \dot{P}_s P_s \]

If \( A \in B(H) \) s.t.
\[ A = PA(H - P) + (H - P)AP, \quad \text{"off-diagonal"} \]

\[ \lim_{\eta \to 0^+} \int_{-\eta}^{\eta} e^{-\imath t} e^{\imath H t} A e^{-\imath H t} \, dt \]

exists. Let \( g(\eta) \) be the spectral gap, then:
(i) \( P \) is a projector:
\[
\left( \frac{-1}{\omega i} \right)^n \int \frac{1}{(H-t)^{-1}} dt' \int \frac{1}{(H-t')^{-1}} dt
\]
where \( \Gamma' \) lies outside \( \Gamma \).

Use the identity:
\[(H-t)^{-1} (H-t')^{-1} = (t-t')^{-1} \left( (H-t)^{-1} - (H-t')^{-1} \right)\]

To get:
(i) \[-\left( \frac{-1}{\omega i} \right)^n \int \frac{1}{(H-t')^{-1}} \int \frac{1}{(t-t')^{-1}} dt \ dt' = 0\]
\[
= 0
\]

(ii) \[-\left( \frac{-1}{\omega i} \right)^n \int \frac{1}{(H-t)^{-1}} \left( -\frac{1}{\omega i} \right) \int \frac{1}{(t-t')^{-1}} dt' = P\]
\[
= 1
\]

(ii) Let \( \Psi \) be an eigenvector for an eigenvalue \( E \) inside \( \Gamma \).
\[
P \Psi = -\left( \frac{1}{\omega i} \right)^n \int \frac{1}{(H-t)^{-1}} \Psi \ dt = -\left( \frac{1}{\omega i} \right)^n \int \frac{1}{(E-t)^{-1}} \Psi \ dt\]
\[
= \Psi
\]
and \( P \Psi = \Psi \) if the eigenvalue lies outside the contour.
\[ \mathcal{P} \text{ being the spectral projection corresponding to } \zeta. \]

\[ \mathcal{P} = \int \limits_{\zeta} d\mathcal{P}(\lambda), \]

hence

\[ \int_{0}^{\infty} e^{-\eta t} \mathcal{P}A(1-P)e^{-i\zeta t} \eta d\eta = \int \int_{\mathbb{R}^2} e^{-\eta t} e^{-it} \eta d\mathcal{P}(\lambda) A d\mathcal{P}(\mu) \]

\[ = \int \int_{\mathbb{R}^2} \frac{+i}{(2-\mu) + i\eta} \mathcal{P}A(1-P) A d\mathcal{P}(\mu) \]  

\[ \eta > 0 \]

because \( A \), the gap \( 1-\mathcal{P} > 0 \) on the support of \( \mu \) is integrable, so that the limit \( \eta \to 0^+ \) is well-defined.

The same holds for \( P \to \mathbb{H} \).

**5.** Let \( \mathcal{J}(A) = \lim_{\eta \to 0^+} \int_{0}^{\infty} e^{-\eta t} \mathcal{T}(A) \eta d\eta \), \( A \) self-adjoint.

Then:

\[ A = -i \left[ \mathcal{H}, \mathcal{J}(A) \right]. \]

namely, the map \( \mathcal{J} \) is the inverse of \( -i \left[ \mathcal{H}, \cdot \right]. \)

Indeed: From (4):

\[ -i \left[ \mathcal{H}, \mathcal{J}(A) \right] = \int \frac{1}{\eta - \mu} \left[ \mathcal{H}, d\mathcal{P}(\lambda) A d\mathcal{P}(\mu) \right] \]

\[ = (2-\mu) d\mathcal{P}(\lambda) A d\mathcal{P}(\mu) \]

\[ = PA(1-P) + (1-P) A P. \]
We are now ready to prove the stochastic theorem.

Let \( R_\varepsilon(s) := \mathcal{P}_\varepsilon(s) - \mathcal{P}_s \).

Denote \( L_s := -i \{ \mathcal{H}_s, \cdot \} \) the generator of \( \mathcal{P}_s \).

Namely,

\[
\varepsilon \dot{\mathcal{P}}_\varepsilon(s) = L_s \mathcal{P}_\varepsilon(s)
\]

and note that \( \mathcal{P}_s \in \ker(L_s) \).

The solution flow \( \mathcal{T}_\varepsilon^{s, s'}(t) \) is given by

\[
\mathcal{T}_\varepsilon^{s, s'}(t) = U_\varepsilon(s, s') \circ U_\varepsilon(s, s')^*
\]

where \( U_\varepsilon(s, s') \) is the propagator associated to \( \mathcal{H}_s \).

Now:

\[
(\varepsilon \frac{d}{ds} - L_s) R_\varepsilon(s) = L_s \mathcal{P}_\varepsilon(s) - \varepsilon \dot{\mathcal{P}}_\varepsilon(s) - L_s \mathcal{P}_\varepsilon(s) + L_s \mathcal{P}_s
\]

\[
= -\varepsilon \dot{\mathcal{P}}_\varepsilon(s)
\]

with, by symmetry, \( R_\varepsilon(0) = 0 \).

This inhomogeneous linear differential equation can be solved explicitly by Duhamel's Principle:

\[
R_\varepsilon(s) = -\int_0^s \mathcal{T}_\varepsilon^{s, s'}(\mathcal{P}_{s'}) ds'
\]

(4a)

Indeed,
\[ \varepsilon \frac{d}{ds} R_\varepsilon(s) = -\varepsilon \dot{P}_s + \int_0^s L_s \left( \sigma_\varepsilon^{S_1S_2} (\dot{P}_s t) \right) dt' \]

\[ -\varepsilon \dot{P}_s + L_s R_\varepsilon(s). \]

Now, if \( \dot{P}_s \) being off-diagonal, \( L_s \) is invertible on \( \dot{P}_s \) and \( \dot{P}_s = L_s L_s^{-1} \dot{P}_s \).

Since \( \sigma_\varepsilon^{S_1S_2} \sigma_\varepsilon^{S_1S_2} = \text{id} \), we conclude that

\[ \varepsilon \left( \frac{d}{ds} \right) \sigma_\varepsilon^{S_1S_2} (A) + 3 \varepsilon \sigma_\varepsilon^{S_1S_2} \left( \frac{d}{ds} \right) \sigma_\varepsilon^{S_1S_2} (A) = 0 \]

namely

\[ \sigma_\varepsilon^{S_1S_2} \left( L_s^{-1} \sigma_\varepsilon^{S_1S_2} (A) \right) \]

\[ \varepsilon \frac{d}{ds} \sigma_\varepsilon^{S_1S_2} (B) = -\sigma_\varepsilon^{S_1S_2} \left( L_s^{-1} B \right) \]

Together, we obtain that

\[ R_\varepsilon(s) = \varepsilon \int_0^s \frac{d}{ds'} \left( \sigma_\varepsilon^{S_1S_2} (L_s^{-1} \dot{P}_s) \right) ds' \]

\[ -\varepsilon \int_0^s \sigma_\varepsilon^{S_1S_2} \left( \frac{d}{ds'} L_s^{-1} \dot{P}_s \right) ds' \]

and hence,

\[ P_\varepsilon(s) - P_s = \varepsilon \left( L_s^{-1} \dot{P}_s - \sigma_\varepsilon^{S_1S_2} (L_s^{-1} \dot{P}_s) \right) \]

\[ -\varepsilon \int_0^s \sigma_\varepsilon^{S_1S_2} \left( \frac{d}{ds'} L_s^{-1} \dot{P}_s \right) ds' \]
This concludes the proof since
\[ \sup \| L^{-1} \dot{P} \| < \infty, \]
\[ \| \int e^{s \gamma} (L^{-1} \dot{P}) \| = \| L^{-1} \dot{P} \| < \infty. \]
\[ \| \int_0^s \int e^{r \gamma} \left( \frac{d}{ds} L^{-1} \dot{P} \right) \| \leq \| \frac{d}{ds} L^{-1} \dot{P} \| < \infty. \]

It remains to prove the claim:

1. \[ \| \dot{P} \| \leq \frac{\| \ddot{P} \|}{q}. \] Indeed: \( \dot{P} \) being self-adjoint,

2. On page 4.3 yields

\[ \dot{P}(1-P) = \frac{1}{2\pi i} \int \int \int \frac{1}{z-w} dP(z) \bar{H} dP(w) \]

and since \( z_0 = 1 \) is the only singularity of the integrand,

\[ \dot{P}(1-P) = \int \int \frac{1}{z-w} dP(z) \bar{H} dP(w) \]

so that since \( |z-w| \geq \gamma \)

\[ \| \dot{P} \| \leq \frac{\| \ddot{P} \|}{q}. \]

(2) On the set of all-adjoint elements, \[ \| L^{-1} \| \leq \frac{1}{q}. \]

This is a very similar calculation:
\[ L_i^\ast (P_{A_i}(1-v)) = \int \int \frac{i}{\lambda - \lambda_i} \, dP_i(\lambda) dP_i(\mu) \]

\[ \|L_i^\ast A\| \leq \frac{\|A\|}{g} \]

3. In particular, \( \sup_s \|L_i^\ast \hat{P}\| \leq \frac{\|A\|}{g^2} < \infty \).

4. Similar estimates hold for the last term, however involving \( \|H_i\| \).

Remarks: 1. Extending \( P_{A_i}(1-v) \) in operator norm is crucial as it allows us to avoid dealing with the remaining dynamics \( \mathcal{S}^\bullet \) over times of order \( \varepsilon^{-1} \).

2. There is a catch: Applying this to a Q.I.S. in a volume \( \Lambda \), where

\[ H^\wedge = \sum \Phi(X) \]

\( X \subset \Lambda \)

is typically expensive, i.e., \( \|H^\wedge\| \sim \|V\| \)

(except if the Hamiltonian changes only in some fixed finite set \( \mathcal{E} \)), the bound is useless.

The regime \( |\|\| \leq \varepsilon^1 \)
I know that the driving rate $\varepsilon \to 0$ as $\varepsilon \to 0^+$. We see later.

If we consider $\mathcal{N}_\varepsilon(s)$ rather than the projector $P_\varepsilon(s)$, then it remains to determine its phase.

By the above, we know that

$$\lim_{\varepsilon \to 0^+} \mathcal{N}_\varepsilon(s) = \mathcal{N}_s$$

exists, and $P_s \psi_s = \psi_s$, namely

$$H_s \psi_s = 0.$$

We define $\mathcal{N}_s : \mathcal{C}^1([0,1]) \to \mathcal{C}^1([0,1])$ by parallel transport condition.

$$\langle \psi_s, \psi_s \rangle = 0.$$ 

Proof: For any $\psi \in C^0([0,1])$, 

$$\int f(s) \langle \psi_s, \psi_s \rangle \, ds = \lim_{\varepsilon \to 0^+} \int f(s) \langle \mathcal{N}_\varepsilon(s), \psi_s \rangle \, ds$$

$$= \lim_{\varepsilon \to 0^+} \int f(s) \langle \psi_s(s), \psi_s \rangle \, ds$$

$$- \lim_{\varepsilon \to 0^+} \int f(s) \langle \mathcal{N}_\varepsilon(s), \psi_s \rangle \, ds$$

First term

$$- \langle \mathcal{N}_\varepsilon(s), \psi_s \rangle = \frac{i}{\varepsilon} H_s \mathcal{N}_\varepsilon(s)$$

so that

$$- \langle \mathcal{N}_\varepsilon(s), \psi_s \rangle = - \frac{i}{\varepsilon} \langle \mathcal{N}_\varepsilon(s), H_s \psi_s \rangle = 0.$$
Second term: Since \( \Phi \in C_c((0,1)) \),
\[
\lim_{\epsilon \to 0^+} \int_0^1 \Phi(s) \langle \Phi(s), \psi_s \rangle ds = \int_0^1 \Phi(s) \| \psi_s \|^2 ds = 0.
\]
Hence
\[
\int_0^1 \Phi(s) \langle \Phi(s), \psi_s \rangle ds = 0 \quad \forall \Phi \in C_c((0,1))
\]
which implies that \( \langle \psi_s, \psi_s \rangle = 0 \).

A bit of geometric jargon: For any \( \sigma \in \mathcal{U} \), there is a one-dimensional subspace of \( H \) given by the ray of \( \sigma \). This is a line bundle of \( H \).

The lemma (usually the Schrödinger equation) provides a way to pick a particular \( \psi_0 \) in the ray of \( \sigma \) by integrating the initial \( \sigma_0 \) along the curve to \( \psi_0 \).

The natural question is then: For a cyclic choice, namely \( H_{\sigma(0)} = H_{\sigma(1)} \)
\( \psi_0 \) is parallel to \( \psi_0 = \omega_0 \), but what is its phase?

Let \( \eta(s) \) be such that \( \Phi(s) = e^{i \eta(s)} \omega_0 \). Then:

\[
\text{Lemma 2: } \eta(1) = -\int 2 \text{Im} \left< \frac{\partial}{\partial s_0}, \frac{\partial}{\partial s_0} \sigma_0 \right> ds_0 dt_0
\]

(but here for \( \mathcal{U} \), 2-dim). In terms of local coordinates,
\[ \dot{\Psi}_s = 0 = \langle \dot{\Psi}_s, \Psi_s \rangle = \langle \dot{\Omega}_s, \dot{\Omega}_s \rangle + i \ddot{\eta}(s) \]

Hence:
\[ \ddot{\eta}(s) = \int_0^1 i \langle \dot{\Omega}_s, \dot{\Omega}_s \rangle \, dr \]

\[ = \int_0^1 i \langle \dot{\Omega}_s \times \Omega_s, \dot{\Omega}_s \times \Omega_s \rangle \, dr \]

\[ \because \int_0^1 \left( \int \langle \dot{\Omega}_s, \partial \Omega_s \rangle - \partial \langle \Omega_s, \dot{\Omega}_s \rangle \right) \, dr = 0 \]

\[ \because \int_0^1 \left( \frac{1}{2} \int \left( \partial \langle \Omega_s, \dot{\Omega}_s \rangle - \langle \dot{\Omega}_s, \partial \Omega_s \rangle \right) \, dr \right) \, d\Omega_s \, d\Omega_s \]

\[ \Delta \]

- Formula: \( \ddot{\eta}(s) \) is called **Berry's phase**. Interestingly, the connection to form \( i \langle \vec{d}l, \vec{d}\Omega \rangle \) has a physical interpretation we see quantum Hall effect, later.

Another view on the quasiparticle transport condition. We have
\[ \Psi_s \dot{\Psi}_s = \dot{\Psi}_s \]

Hence by differentiation,
\[ \dot{\Psi}_s \dot{\Psi}_s + \dot{\Psi}_s \dot{\Psi}_s = \ddot{\Psi}_s \]

\( (\dot{\Omega}_s - \dot{\Omega}_s) \dot{\Psi}_s = \dot{\Psi}_s \dot{\Psi}_s \)

As the condition that \( \dot{\Psi}_s \) remains in \( \text{Re} \, \Psi_s \) for all \( s \) determines only the \( (\dot{\Omega}_s - \dot{\Omega}_s) \) component of \( \dot{\Psi}_s \).

In \( P \)-transport:
\[ \Psi_s \dot{\Psi}_s = 0 \], no charge within \( \text{Re} \, \Psi_s \).
Locality and parallel transport.

1950: Kato:

\[ \psi_s = \psi_s + (M - \mathcal{P}_s)\psi_s = \psi_s \psi_s = (\mathcal{P}_s - \mathcal{P}_s) \hat{\psi}_s \psi_s \]

\[ = [\hat{\psi}_s, \mathcal{P}_s] \psi_s \]

Hence, one can recast parallel transport as a unitary propagation:

\[ \psi_s = U^k(s, 0) \psi_0 \]

where \( U^k(s, s') \) is the propagator given by

\[ i \frac{d}{ds} U^k(s, s') = i [\hat{\psi}_s, \mathcal{P}_s] U^k(s, s') \]

\[ U^k(s, s') = e^{-i s \mathcal{P}_s} \]

And the adiabatic theorem can be phrased as comparing \( U^k(s, s') \) with Kato's propagator \( U^k(s, s') \).

Now: Brand L Q.S.P.:

\[ H_{s, \lambda} = \sum_{X \in \Lambda} \lambda X \]

a special gap, uniformly in \( \lambda \) \( \leq [0, 1] \) and \( \Lambda \).

Observe: Since \([\hat{\psi}, \mathcal{P}]\) has no natural structure as a sum of local terms, \( U^k(s, s') \) has no reason to satisfy L.H.B. and hence

\[ U^k(s, 0)^\Lambda A U^k(s, 0) = U^k(s, s') A U^k(s, 0) \]

has no reason to be small for a generic local \( A \).
Hashing's insight (2004): The $(A-P_i)(1/A-P_i)$ block of the generator can be chosen at will without modifying the $L^2$-transport property.

**Assumptions.** A.S.S. will $\Phi_s \in B_{\Phi_2}$ $(\alpha > 0)$
for all $s \in [0,1]$,

1. For any $X: \Phi_s(X)$ is $C^1$.
2. $\partial \Phi_s(X) := \{X|\Phi_s(X)\}$ denotes an intersection and $\partial \Phi_s \in B_{\Phi_2}$ $(\alpha > 0)$
   for all $s \in [0,1]$.

**Theorem:** There exists a function $W \in L^1(M; \mathbb{R})$ such that

$$G_s := \int_0^1 \langle \Phi_t \rangle e^{-\alpha t} s \int_0^{s'} e^{-\alpha t'} dt'$$

and $U^s(s,s')$ is the corresponding propagator, then:

1. $P_s = U^s(s,s') P_{s'} U^{s'}(s,s')$
2. The $\omega$-automorphism $\alpha s, s$ $W := U^s(s,s') A U^{s'}(s,s')$

satisfies a Lieb-Robinson bound.

Moreover, $W$ can be chosen to be odd and so that

$$\langle (1+i\xi)^n \rangle (W(t)) \in L^1 \text{ for } \xi \geq 1.$$

**Remarks.** (ii) implies in turn that the automorphism extends to the limit $\lambda \to \lambda$, in which.
We obtain the following:

Let $S_\omega = \{ a_\omega ; a_\omega \text{ is a state on } \mathcal{A} \}
\quad \text{and } a_\omega = \lim_{n \to \infty} \langle \eta_\omega^{(n)} , \mathcal{A} \eta_\omega^{(n)} \rangle
\quad \text{where } \mathcal{P}_\eta^{(n)} \eta_\omega = \eta_\omega^{(n)} , \| \mathcal{P}_\eta^{(n)} \| = 1

(\text{the limit does not necessarily exist, but there is a convergent subsequence}). Then

$S_\omega = S_\omega \circ \alpha_{s,0}^\omega$

uniquely $\alpha_{s,0}^\omega$ establishes a bijection between the two sets, where $\omega_s = \alpha_{s,0}^\omega \circ \alpha_{s,0}^\omega
\quad \text{a W decay} \text{ faster than any inverse power as } |t| \to \infty,$

but it cannot be exponentially decaying.

First, we show that the map

$$T_{W}(f) = \int_{-\infty}^{\infty} e^{itf} e^{-tf} \, df$$

(for a suitable choice of $W$) is again an inverse of $i[s,\mathbb{I}]$ on the set of all-optimal operators.

For any $W \in L^1$: $\| T_{W}(f) \| \leq \int |W(t)| \| A \| \, dt
\quad = \| W \|_{L^1} \| A \|$

Since $W \in L^1$, its Fourier transform

$$\hat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(t) e^{-it\xi} \, dt$$
is well-defined.

Assume that \( \hat{W}(\xi) = \frac{-i}{|\xi|^3} \) whenever \( |\xi| > g \)

where \( g \) is the \( g \)-isochron \( \Phi \). Then

\[
-i [H_s, \Gamma^s_w(A)] = -i \int dt \, W(t) \left[ H_s, e^{itH_s} A e^{-itH_s} \right]
\]

\[
= -i \int_0^\infty dt \, W(t) \int (2-\mu) e^{i(2-\mu)} d\nu \, d\mu \, d\nu (A) d\mu (A)
\]

\[
= -\sqrt{\mu} \int_0^\infty \hat{W}(\mu-2)(2-\mu) d\nu (A) d\mu (A)
\]

and the integral extend only to \( |2-\nu| > g \) by the \( A \)-isochronity assumption on \( A \). Hence

\[
-i [H_s, \Gamma^s_w(A)] = \int \left[ \sqrt{\mu} \frac{-i}{\sqrt{\mu}} (\mu-2) d\nu (A) d\mu (A) \right]
\]

\[
= A
\]

as announced.

Proof of theorem. (i) We note that \( G^s_t = \Gamma^s_w(H_s) \). Recalling that \( \hat{P}_s \) is \( \sigma \)-diagonal:

\[
\hat{P}_s = -i [H_s, \Gamma^s_w(\hat{P}_s)] = -i \frac{\Gamma^s_w}{\Gamma^s_w([H_s, \hat{P}_s])}
\]

Since \( [H_s, \hat{P}_s] = 0 \), we have

\[
[H_s, \hat{P}_s] = [\hat{P}_s, H_s]
\]

and so.
\[ \Phi_s = +i T^s_w \left( [\hat{H}_s, P_s] \right) \]
\[ = i \left[ T^s_w (H), P_s \right] \quad \text{(again because } [H, P] = 0) \].

This concludes the proof of (i) by uniqueness of the projector.

(ii) (Sketch) It suffices to show that the generator \( C_t^{\overline{H}} \) is a sum of interaction in one of the bounded space \( B_p \)

\[ C_t^{\overline{H}} = \sum_{\text{exc}} \int_0^T \langle \overline{W}(t) \rangle e^{i t H} \Phi(X) e^{-i t \overline{H}} \, dt \]

where \( \Phi(X) \in A \). We show that if \( A \in A \), then \( T^s_w (A) \) is approximately local. By the LRB of \( \tau \),

\[ \| [e^{i t \overline{H}} A e^{-i t \overline{H}}, Z] \| \leq C \| A \| \| X \| e^{\Delta t} e^{-\Delta t} \| Z \| \]

for all \( Z \in A_x \) (i.e. dist \( (X, \text{sup} \| A \|) > \Delta t \)). Hence there exists \( \tilde{A}_t \in A \) such

\[ \| e^{i t \overline{H}} A e^{-i t \overline{H}} - \tilde{A}_t \| \leq C \| A \| \| X \| e^{\Delta t} e^{-\Delta t} \]

and hence

\[ \| \int_0^T \langle \overline{W}(t) \rangle e^{i t \overline{H}} A e^{-i t \overline{H}} - \tilde{A}_t \rangle \, dt \| \]

\[ \leq 2 \| \overline{W} \|_{\infty} C \| A \| \int_0^T e^{\Delta t} e^{-\Delta t} \, dt \]

while
\[ \left\| \int_{t>T} |W(t) (e^{itH}Ae^{-it\tilde{H}} - A_t)dt \right\| \leq 4 \|A\| \int_{t>T} |W(t)|dt. \]

Since \( W(t) = O(t^{-m}) \), so \( \int_{t>T} |W(t)|dt \) tends to zero. It remains to pick
\[ T = \frac{n}{2v} \]

to get:
\[ \left\| \int_{t>T} |W(t) (e^{itH}Ae^{-it\tilde{H}} - A_t)dt \right\| \leq C_n \left( \frac{1}{n} \right) + C e^{-\frac{an}{2}} \]

This shows that \( W(A) \) can be well approximated by an operator supported in \( A_{x_i} \) for \( n \in \mathbb{N} \).

The rest of the proof is a tedious transcription of each term in \( G_s^H \) and reorganization of terms to write
\[ G_s^H = \sum_{X \in \mathcal{A}} \Psi_s^H(X) \]

where \( \Psi \in \mathcal{B} \), where \( \Psi \) decays faster than any inverse power. Hence the dynamics generated by \( G_s^H \) satisfy uniform uniformity in \( s \in [0,\tau] \). \( \square \)

We assume that the point \( \Sigma_s \) is in fact just one point corresponding to a possibly degenerate eigenvalue \( E_0 \).

Then,
\[ P_s G_s^H P_s = \int_{t>T} e^{itH} P_s H s P_s e^{-itE_0} dt = \tilde{W}(0) P_s H_s P_s. \]

Set \( \omega = 0 \) since \( W \) is an odd function.
We conclude that for any $\Psi_0 = P_0 \Psi_0$

$$P_0 \frac{d}{dt} U(t, s) \Psi_0 = -i P_0 G^{\dagger} U(t, s) \Psi_0 = -i P_0 G^{\dagger} P_0 \Psi_0 = 0$$

namely, the vector $U(t, s) \Psi_0$ satisfies the boundary transport condition.

but by changing the generator to $G^{\dagger}$, the dynamics on the full Hilbert space has become local, as if it were adiabatic evolution.

Remark: $U(t, s')$ is sometimes called the "quasi-adiabatic" evolution.

We now have the tools to prove the many-body adiabatic theorem. The setting is that of a D.S. (or lattice fermions).

Assumptions: (1) Gap: $\text{Spec}(H_s) \subset \mathbb{Z}_s^0 \cup \mathbb{Z}_s^0$ with

$$\inf \{ \text{dist}(\mathbb{Z}_s^0, \mathbb{Z}_s^0) \} = g > 0$$

uniformly in the volume.

(2) Interaction, $H_s = \sum_{X \in \Lambda} \Phi_s(X)$ where $\Lambda \subset \mathbb{Z}^d$.

$s \mapsto \Phi_s(X)$ is $C^{d+1}$.

a The interactions defined by $|X|^n \Phi_s^{(n)}(X)$ all belong to $B_{G_0}$, where $a = a(n, k), 0 \leq k \leq d+1$.

Example: nearest-neighbour interaction, with a pairwise interaction that depends smoothly on $s$. 

Let \( \Psi_\varepsilon(s) \) be the solution of

\[
\begin{align*}
1 & \in \Psi_\varepsilon(s) = H_s \, \Psi_\varepsilon(s), \\
P_s \, \Psi_\varepsilon(0) = \Psi_\varepsilon(0), \quad \|\Psi_\varepsilon(0)\| = 1
\end{align*}
\]

and \( P_s \) is the spectral projector of \( H_s \) for \( \varepsilon^{(1)} \).

**Theorem:** For \( s \in [0,1] \), there is \( \omega_\varepsilon(s) \in \mathbb{R} \cup \{ \infty \} \), \( \| \omega_\varepsilon(s) \| > 1 \),

\[
\left| \langle \Psi_\varepsilon(s), O \Psi_\varepsilon(s) \rangle - \langle \omega_\varepsilon(s), O \omega_\varepsilon(s) \rangle \right| \leq C \left( \sup_{\Omega} \| O \| \right) \varepsilon
\]

for any local observable \( O \).

**Remark:** If the interaction is \( C^0 \)-smooth, and \( \Psi_\varepsilon(x) \) is constant in a neighbourhood of \( s = 0 \) and \( s = 1 \), then the bound holds at \( s = 0 \) with \( \varepsilon^m \).

If \( \Delta = \sup_{s} \left( \max_{x} \varepsilon^{(1)}_s - \min_{x} \varepsilon^{(1)}_s \right) \), then

\[
\Delta \leq C \min \left( \varepsilon^2, \frac{\varepsilon}{|\Omega|} \right)
\]

(for example, one degenerate eigenvalue), then \( \omega_\varepsilon(s) \) can be chosen to be parallel transported:

\[
\omega(s) = \Psi_0, \quad P_s \omega_s = 0
\]

in which case the choice is \( \varepsilon \)-independent.

The case of non-interacting spin \( \Phi_\varepsilon(X) = 0 \) if \( |X| \neq 1 \) and an initial state \( \Psi_0 = \bigotimes_{x \in \Omega} \psi_x \).
where $\Psi_x$ is the unique ground state of $\Phi_0(\phi x_1)$. Then: (i) the spin evolve independently:

$$\Psi_x(s) = \bigotimes_{x \in \Lambda} \Psi_{x,s}(s)$$

(ii) the adiabatic dec. (in standard form) applies at each site.

$$\left< \Psi_{x}(s), \bigotimes_{x \in \Lambda} \Psi_{x,s} \right> = \prod_{x \in \Lambda} \left< \Psi_{x}(s), \Psi_{x,s} \right> = 0(3)$$

is of order $3^n$.

The true evolved state and the instantaneous ground state are orthogonal to the infinite value limit: "inferred catastrophe" or "orthogonally infinite".

The catastrophe is a reality, but it doesn't affect local observations.

Proof of the many-body adiabatic theorem:

First a notation. $A \in \mathbb{R}^\Lambda$ is $A = \sum_{x \in \Lambda} \Phi(x)$ with

$$\sup_{x \in \Lambda} \sum_{x \in \Lambda} |x|^n \frac{\|\Phi(x)\|}{F(d(x, y))} < \infty$$

for all $N \in \mathbb{N}$ and on $F$: $\sup |(4\pi)^4 F(r)| < \infty$. 
\[ V \cap \bigcup_{j} U_{j} = \emptyset \]

and \( U_{j} \cap U_{i} = \emptyset \) for \( i \neq j \), hence \( U = \bigcup_{j} U_{j} \). Let \( F = \bigcap_{j} U_{j} \). Then \( F \subseteq U \cap \bigcup_{j} U_{j} \).

Therefore, if \( G = U \cap \bigcup_{j} U_{j} \), we have \( G = F \).

Read the following:

For instance, \( \exists x \in \mathbb{R}^3 \), which is close to the ground state. If \( \exists x \in \mathbb{R}^3 \), it will be selected. Now, consider

where \( R \subseteq \mathbb{R}^3 \) contains \( \exists x \in \mathbb{R}^3 \). Then

\[ \lim_{s \to 0^+} \frac{1}{s} \int_{\mathbb{R}^3} R^{\frac{3}{2}} \| F(s) \|_{L^2} \, dV \rightarrow 0 \]

for all \( s \in \mathbb{R}^3 \). The limit is uniform, and thus

\[ \lim_{s \to 0^+} \frac{1}{s} \int_{\mathbb{R}^3} R^{\frac{3}{2}} \| F(s) \|_{L^2} \, dV = 0 \]

with

\[ \left( -\frac{1}{A} \right)^{\frac{3}{2}} \frac{1}{|s|} \left( \frac{1}{s} \right)^{\frac{3}{2}} \int_{\mathbb{R}^3} R^{\frac{3}{2}} \| F(s) \|_{L^2} \, dV \]

where \( G \subseteq \mathbb{R}^3 \). There exist \( \exists x \in \mathbb{R}^3 \) such that

\[ \lim_{s \to 0^+} \frac{1}{s} \int_{\mathbb{R}^3} R^{\frac{3}{2}} \| F(s) \|_{L^2} \, dV = 0 \]

Please ignore the projection.
\[ i \epsilon \hat{\Pi} = \left[ H, \hat{\Pi} \right] + \left[ i \epsilon \hat{U} \hat{U}^* - \epsilon U U^*(H) U^* + (U H U^* - H), \Pi \right] \]

where we noted that \[ \left[ U H U^*, \Pi^* \right] = 0 \] and the inner term is \( n \) dimensional.

Second commutator:

\[ U \left[ i \epsilon \hat{U} \hat{U}^* - \epsilon U U^*(H) + (H - U H U^*), \Pi \right] U^*. \tag{8} \]

Goal: choose \( \Pi \) recursively so as to cancel out the increasing order of \( \epsilon \) in the commutator.

Recall \( \hat{U} = \exp \left( i 2 \hat{S} \right) \), \( \hat{S} = \sum_{\alpha} \epsilon \hat{A}_\alpha \).

Then \( \hat{U}^\dagger \hat{H} \hat{U} = \hat{H} + \sum_{k=0}^\infty \frac{(i \epsilon)^k}{k!} \left( \sum_{\alpha} \epsilon \hat{A}_\alpha \right)^k \hat{H} \)

where \( \text{ad}_H(N) = [H, N] \)

(Indeed, \( \hat{U}^\dagger \hat{H} \hat{U} = e^{-i \text{ad}_{\hat{S}}(\hat{H})} \).

\[ = \hat{H} + \sum_{k=1} \frac{\epsilon^k}{k!} \hat{H}_k + \epsilon^k \hat{h} \]

where \( \hat{H}_k = \sum_{\|j\|_h = \alpha} \frac{(i \epsilon)^k}{k!} \text{ad}_{A_{j_1}} \cdots \text{ad}_{A_{j_k}} (H) \)

namely \( \hat{H}_k = -i [A_k, H] \)

\[ \hat{H}_k = -i [A_k, H] - \frac{1}{2} [A_k, [A_k, H]] \]

...
Do the same with
\[ \text{ii} \]
where
\[ \phi = \int e^{-i H t} \hat{S}_e \hat{d} \hat{s} d^2 \]
\[ = - \sum_{\alpha=1}^d \sum_{\epsilon=1}^{d+1} \frac{(-1)^\epsilon}{\epsilon} \frac{\partial}{\partial \lambda_{k_{\alpha}}} \delta_{\hat{A}_{j_{\epsilon}}} (\hat{A}_{j_{\epsilon}}) \]

Here:
\[ \phi_{\alpha} = - i \sum_{\epsilon=1}^{d+1} \frac{\partial}{\partial \lambda_{k_{\alpha}}} \delta_{\hat{A}_{j_{\epsilon}}} (\hat{A}_{j_{\epsilon}}) \]

\[ \hat{A}_{\alpha} = - \hat{A}_{\alpha} \quad \hat{a}_{\alpha} = - \hat{A}_{\alpha} + \frac{i}{2} [\hat{A}_{\alpha}, \hat{A}_{\alpha}] \]

Plug in (\(\phi\)) which has no zeroth order:

\(x = \hat{a}_n \quad y = \hat{a}_n + \frac{i}{2} [\hat{a}_n, \hat{a}_n] \)

\([\hat{T}_\omega (\hat{H}) - i [\hat{A}_n, \hat{H}], \hat{P}]] = 0 \]

Claim:
\[ \hat{A}_n = \hat{T}_\omega (\hat{T}_\omega (\hat{H})) \]

is a solution of this equation.

Proof: Since \([L, \hat{P}] = 0\) for any \(L\), we know that

\[-i \left[ \hat{H}, \hat{T}_\omega (\hat{L}, \hat{P}) \right] = \left[ \hat{L}, \hat{P} \right] \]

\[\hat{T}_\omega (\hat{L}, \hat{P}) = \left[ \hat{T}_\omega (\hat{L}), \hat{P} \right] \]

\[\left[ \exp i \phi (\hat{L}, \hat{P}) \right] = - \left[ \hat{T}_\omega (\hat{L}), \left[ \hat{P}, \hat{H} \right] \right] - \left[ \hat{P}, \left[ \hat{H}, \hat{T}_\omega (\hat{L}) \right] \right] = 0 \]
Usually: 
\[ \left[ i \left[ Tw(L), H \right] - L, P \right] = 0 \]

conclude by setting \( L = Tw(H) \).

Order 2: 
\[ \left[ i (Q_1 + H_2), P \right] \]

\[ \left[ -i A_1 - i \left[ A_1, H \right], \frac{1}{i} \left[ A_1, \left[ A_1, H \right] \right], P \right] = 0 \]

which is solved by the same which:

\[ A_2 = Tw \left( -i A_1 - \frac{1}{i} \left[ A_1, \left[ A_1, H \right] \right] \right) \]

given that \( H \) exists (because it appears in \( A_1 \)).

Next order: The next order already appears as

\[ -i \left[ A_2, H \right] \]

so \( Tw \) yields a solution.

Since \( H \in C^{d+1} \), the construction stops with \( A_{d+1} \).

Note: The whole construction is local:

\[ A \in L_2 \iff Tw(A) \in L_2 \]

\[ A_1, A_2 \in L_2 \iff \left[ A_1, A_2 \right] \in L_2 \]

The proof of the theorem is now standard.

Let \( \psi_{e}(s) = U_{e}(s,t) \phi_{0} \) be the time-evolved state.

Let \( \psi'_{e}(s) = V_{e}(s) \psi_{e}(s) \) a "dressed state."

Then:

\[ \left< A \psi'_{e}(s), \psi'_{e}(s) \right> - \left< \psi'_{e}(s), A \psi'_{e}(s) \right> \]

\[ = \text{Tr} \left( U_{e}(s,t) P_{e} U_{e}(s,t) A \right) - \text{Tr} \left( U_{e}(s) P_{e} V_{e}(s)^{\dagger} A \right) \]
Note: \( U_\varepsilon(s,0) P_\varepsilon U_\varepsilon(s,0)^* - U_\varepsilon(s) P_\varepsilon U_\varepsilon(s)^* \bigg|_{r=0}^{r=S} \)

Hence: \( \langle \bar{N}_\varepsilon(s), A\bar{N}_\varepsilon(s) \rangle - \langle \psi_\varepsilon(s), A\psi_\varepsilon(s) \rangle \)

\[
= - \int_0^S \frac{d}{dr} \text{Tr} \left( U_\varepsilon(s,r) \bar{P}_\varepsilon(r) U_\varepsilon(s,r)^* A \right) dr
\]

\[
= - \frac{i}{\varepsilon} \int_0^S \text{Tr} \left( U_\varepsilon(s,r) \left[ H_r - (H_r + R_\varepsilon(r)), \bar{P}_\varepsilon(r) \right] U_\varepsilon(s,r)^* A \right) dr
\]

\[
= \frac{i}{\varepsilon} \int_0^S \text{Tr} \left( U_\varepsilon(s,r)^* A U_\varepsilon(s,r), R_\varepsilon(r) \right) \psi_\varepsilon(r) \rangle dr
\]

Now: \( \text{Lemma } \| [U_\varepsilon(s,r)^* A U_\varepsilon(s,r), R_\varepsilon(r)] \psi_\varepsilon(r) \rangle \|

\leq C |\text{supp } (\eta)| \|A\| \varepsilon^{-d} \| \Phi_{R_\varepsilon} \|_F
\)

Proof: see exercises.

Recall: by construction, \( \| \Phi_{R_\varepsilon} \|_F \leq C \varepsilon^{d+2} \), here

\[
\left| \langle \psi_\varepsilon(s), A\psi_\varepsilon(s) \rangle - \langle \psi_\varepsilon(s), A\bar{N}_\varepsilon(s) \rangle \right| \leq C(A) \varepsilon^{-1-d+d+2}
\]

It remains to compare \( \psi_\varepsilon(s) \) with the instantaneous ground state

\[
\langle \psi_\varepsilon(s), A\psi_\varepsilon(s) \rangle - \langle \psi_\varepsilon(s), A\bar{N}_\varepsilon(s) \rangle = \langle \psi_\varepsilon(s), e^{-i \mu S_{\varepsilon}(s)} A e^{i \mu S_{\varepsilon}(s)} \bar{N}_\varepsilon(s) \rangle
\]

\[
= i \int_0^S \left[ e^{-i \mu S_{\varepsilon}(s)} A e^{i \mu S_{\varepsilon}(s)} \right] \psi_\varepsilon(s) \rangle dr
\]

evolution for times of order \( 1 \)

here (see lemma). \( \| \psi_{S_{\varepsilon}(s)} \|_F \leq C \varepsilon \).
Note: If \( H^{(s)}(s=1) = 0 \) \( 1 \leq c \leq d+2 \), then 
\[
A_0(s=1) = 0 \quad (\forall \alpha) \quad \text{and hence} \quad c_0(s=1) = 0
\]
It follows that the dressing channel vanishes at \( s=1 \)
is the identity \( c_0^{(s)} = 0 \). The error is
bounded by \( e_{\alpha}^m \) if \( H \) is \( d+m \times d+m \)
because one can construct \( A_0 : 1 \leq \alpha \leq d+m \).

We now move to one application of both the adiabatic theory and the quasi-adiabatic flow: Linear response theory.

**Setting:** Assign Hamiltonians \( H_i \) of a quantum spin system.

1. Slow switching of a driving \( V \) over \((0,0]\): 
   \[
   H_i = H_i + \alpha e^{\epsilon t} V \quad (0 < \alpha, \epsilon \ll 1).
   \]

2. Question: What is, to lowest order in \( \alpha \), the response of the system to the perturbation?

Here: Initial state at \( t \to -\infty \): Ground state projection \( \Pi_i = \frac{1}{\text{dim}(\Pi_i)} \Pi_i \)

- Observable \( \bar{J} \in \mathcal{A}_{loc} \).

Compute:
\[
| \text{Tr}(\bar{g}_{e(0)}(J)) - \text{Tr}(\Pi_i J) | 
\]
where
$\rho_\alpha(t)$ is the time evolved state over $[-\infty, 0] \text{ with time dependent Hamiltonian } H(t)$.

- Typical example: the describer electrons hopping and interacting in a crystal (i.e. a lattice).
- $V$ is a tension that is imposed across a sample (or a general electromotive force).
- $J$ is the electric current along an edge.
- Linear response is the relation
  \[ \langle J \rangle = \sigma \alpha + O(\alpha^2) \]
  with $\alpha$ expression for $\sigma = \sigma(V)$ the conductance.
- Often, $Tr(P_j \cdot J) = 0$ : no steady currents.

The linear response coefficient ("generalized susceptibility") is defined by

\[ X_{\alpha, \alpha'}(\omega) = \lim_{\alpha, \alpha' \to \infty} \lim_{\omega \to \omega_0} \frac{Tr(J \rho_{\omega_0}(t)) - Tr(J \rho_{\omega_0}(t))}{\alpha} \]

The order is important, large volume limit first "microscopic limit".

Assumption: both $H(t)$ and $V$ are given by local interchanges in a Br.
Existence of a state \( \psi \) on the infinite volume \( \mathbb{R}^3 \): A st.

\[
\lim_{\lambda \to 1} \text{Tr} (\overline{\mathcal{P}}_\lambda A) = c_\lambda (A) \quad (A \in \mathcal{A}).
\]

Uniform spectral gap for \( H_\lambda + \beta V \) for all \( \lambda \in (-\delta, \delta) \) and volumes.

\[\Theta \] see [6x1]

**Theorem:** The limit (x) exists and is given by

\[
X_{\lambda, \nu} = -i \alpha_\lambda \left( [ \tilde{W}_\lambda (U), J] \right). \quad (\text{LRT})
\]

Remark: \( X_{\lambda, \nu} \) is expressed solely in terms of the initial data.

\[\Rightarrow \] lim \( \lambda \to 1 \) can be strengthened to lim \( (\alpha, \nu) \to (0, 0) \).

Formal calculation: Set \( J = H_\lambda \): we compute the change of energy of the system, i.e., the work. Thus

\[
\text{Tr} \left( \overline{\mathcal{P}}_\lambda [\tilde{W}_\lambda (U), H_\lambda] \right) = \text{Tr} \left( [H_\lambda, \overline{\mathcal{P}}_\lambda \tilde{W}_\lambda (U)] \right) = 0.
\]

No work performed, typical for the Lorentz force caused by a magnetic field. See quantum Hall effect.

For \( \nu > 0 \): (LRT) can be merged to a well-known formula in the physics literature: Kubo's formula.
$X_{\gamma, \nu}$ is defined by $\Phi$ where $g_{\gamma, \nu}(t), \nu (\infty, 0)$ is given by

\[
\begin{align*}
\frac{d}{dt} g_{\gamma, \nu}(t) &= \begin{bmatrix} H_l + \alpha e^{zt} V, & g_{\gamma, \nu}(t) \end{bmatrix} \\
\lim_{t \to -\infty} e^{iH_l t} g_{\gamma, \nu}(t) e^{-iH_l t} &= \overline{\Phi_i}
\end{align*}
\]
Proof: For all $t \in (-\infty, 0)$,

$$\text{Rank}(P_t) - \text{Rank}(P_0) = \text{Rank}(P_c)$$

The adiabatic theorem yields

$$|\text{Tr}( (\text{sel}(\alpha) - P_0) J )| < C \alpha \varepsilon$$

(1)

where $C$ depends on the interactions of $H_i, V$ and on $J$ but is independent of $\Lambda$.

Recall the quasi-adiabatic flow

$$\lim_{\Lambda \to \infty} U_{\alpha}^H(\alpha, 0) U^H(\alpha, 0) = \iota_{\alpha, 0}^H(J)$$

such that

$$i \frac{d}{d\alpha} U^H(\alpha, 0) = \iota_{H_i}(V) U^H(\alpha, 0)$$

since

$$\frac{d}{d\alpha} (H_i + \alpha V) = V$$

Thus:

$$\lim_{\Lambda \to \infty} \sup_{\alpha \to 0^+} |\text{Tr}( (\text{sel}(\alpha) - P_0) J )| = 0$$

while

$$\lim_{\Lambda \to \infty} \text{Tr}( (P_i - P_0) J ) = \lim_{\Lambda \to \infty} \text{Tr}( P_i (U_{\alpha}^H(\alpha, 0) U^H(\alpha, 0) - J) )$$

$$= \omega_i ( \iota_{\alpha, 0}^H(J) - J )$$

We conclude by

$$-\lim_{\Lambda \to \infty} \sup_{\alpha \to 0^+} \omega_i ( \iota_{\alpha, 0}^H(J) - J ) = - \frac{d}{d\alpha} \omega_i ( \iota_{\alpha, 0}^H(J) ) |_{\alpha = 0}$$

$$= i \omega_i \left( [J, T_{H_i}(V)] \right)$$

Note that by (1), the order of limits $\alpha, \varepsilon$ is irrelevant, for the first term, as it is a portion for the second,
which is independent of $\epsilon$. Hence,
\[
\lim_{\epsilon \to 0} \lim_{\gamma \to 1} \alpha^{-1} \Gamma \left( \left( \frac{1}{\epsilon}, \frac{1}{\gamma}, 0 \right) - \epsilon \right) = -i \omega \left( \left( \Gamma \omega^{(1)} / \gamma, 0 \right) \right)
\]

Interestingly, or surprisingly, the linear response coefficient is related to Fermi's phase.

System a charged fermions hopping on a lattice $\Gamma$, will Hamiltonian $H$

- coupling to electromagnetic field by choosing the hopping term:
\[
\sum_{\langle x,x' \rangle} \alpha(x,x') \left( a_x a_x^\dagger + a_{x'} a_{x'}^\dagger \right)
\]

to the field $A$
\[
\alpha(x,x') \rightarrow \alpha(x,x') e^{iA(x,x')}
\]

(Strictly speaking, $A(\cdot)$ is defined on edges of $\Gamma$)

- In particular: a slowly varying $A$ will drive a current through the system.

Current out of a set $T$:
\[
\text{Current} \ n_T = \frac{1}{\epsilon} \int_{x \in T} n_x dt
\]

\[
[\partial_t = i \left[ H, n_T \right]
\]

\[
\begin{align*}
\text{Current} & = \frac{1}{\epsilon} \int_{x \in T} n_x dt \\
\text{Charge} & = \int_{x \in T} n_x dx
\end{align*}
\]
which can be obtained by

\[
J_\alpha = \frac{2}{\alpha \phi, \eta} e^{i \phi, \eta} H e^{-i \phi, \eta} \mid \phi = 0
\]

Note: By choosing \( \alpha \) to be a half system, \( J_\alpha \)

is the current across a bidirectional hyperplane.

\[
\alpha \frac{\partial \phi}{\partial \rho} \leq 1
\]

\[
\alpha \frac{\partial \phi}{\partial \rho} \geq 1
\]

- Locally: \( A(x, x') = V(x) - V(x') \), Recall also (Sheet 3)

\[
\alpha \frac{\partial \phi}{\partial \rho} a e^i = e^i a
\]

Hence \( A \) can be implemented using \( V \), locally in

a set \( \mathcal{Z} \):

\[
V_{\xi} = \sum_{x \in \mathcal{Z}} V(x) n_x
\]

\[
e^{i \phi_{\xi} V_{\xi}} \partial_{\xi} x e^{-i \phi_{\xi} V_{\xi}} = e^{-i \phi_{\xi} (V(x) - V(y))} \partial_{\xi} x e^{-i \phi_{\xi} V_{\xi}}
\]

and again:

\[
i [H, V_{\xi}] = \frac{2}{\alpha \phi, \eta} e^{i \phi_{\xi} V_{\xi}} H e^{-i \phi_{\xi} V_{\xi}}
\]

Now: (i) gives the current observable, and (ii) the driving:

\[
\text{Tr} \left( P \left[ T_{\xi}(V), \mathcal{Z} \right] \right) = \text{Tr} \left( \left[ \mathcal{Z}, P \right] T_{\xi}(V) \right)
\]

\[
= i \text{Tr} \left( \left[ \left[ T_{\xi}(V), H \right], P \right] T_{\xi}(V) \right)
\]

and with some algebra:

\[
= i \text{Tr} \left( P \left[ T_{\xi}(V), \left[ T_{\xi}(V), H \right] \right] \right)
\]

\[
= -i \text{Tr} \left( P \left[ T_{\xi}(V), T_{\xi}(H), V \right] \right)
\]

\[
- i \text{Tr} \left( P \left[ H, T_{\xi}(V), T_{\xi}(V) \right] \right)
\]

\[
= 0
\]
In other words:
\[ \chi_{\tau, \nu} = \text{Tr} \left( P \left( \tau_w(\varphi_{\epsilon_1} H), \tau_x(\varphi_{\epsilon_2} H) \right) \right). \tag{7} \]

Remark: \( H(\varphi_1, \varphi_2) \) is obtained by conjugating with \( e^{i\theta_n} \).

And since \( \text{Spec}(\tau) \subset \mathbb{C} \cdot \mathbb{N} \),

so that \( H(\varphi_1, \varphi_2) \) is doubly periodic.

In Family \( H_{\varphi}, \phi \in \mathbb{T} \), the torsion:

\[ \mathbb{T}^2 \]

\[ \varphi \]

\[ \varphi_1 \]

\[ \tau_w(\varphi_{\epsilon_1} H) \]

\[ \tau_x(\varphi_{\epsilon_2} H) \]

are the generators of the transport along two fundamental loops of the torus: hence from (7):

\[ \chi_{\tau, \nu} = \text{Tr} \left( P \left( \tau_w(\varphi_{\epsilon_1} H), P \right), \tau_x(\varphi_{\epsilon_2} H), P \right) \right) \]

\[ = \text{Tr} \left( P \left( \varphi_{\epsilon_1} P, \varphi_{\epsilon_2} P \right) \right) \]

In the case where \( P \) is a one-dimensional projection onto \( \varphi_{\epsilon_1} \), and recalling that \( \langle \varphi, \varphi_{\epsilon_1} \varphi \rangle = 0 \).
\[ X_{j,v} = -i \left( \langle \phi_{j,v}, \phi_{k,v} \rangle - \langle \phi_{j,v}, \phi_{k,v} \rangle \right) \]

\[ = 2 \text{Im} \left( \langle \phi_{j,v}, \phi_{k,v} \rangle \right) \]

This is the **adiabatic curvature**, which yields Berry's phase when integrated over a surface.

Linear response coefficient = adiabatic curvature

(Thouless et al., 82; Avron et al., 85)