The mathematical structure of quantum mechanics.

In QM, the state of a N particles is characterized by specifying their position and momenta, i.e. it is a point in phase space \( \mathbb{R}^{2N} \).

More generally, in the presence of statistical uncertainty, a state is a probability measure \( \mu \) on \( \mathbb{R}^{2N} \).

A state allows one to compute the values of observables (expected).
An observable is a function on phase space \( f : \mathbb{C}^{2N} \rightarrow \mathbb{R}^{2N} \) (the set of continuum function vanishing at \( 0 \)).

E.g., the energy

\[
E(x,p) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \sum_{1 \leq i < j \leq N} \frac{\alpha}{|x_i - x_j|^4}
\]
and given a state:
\[ \mu(f) = \int y \, dm, \]

where
\[ \int \psi(x_1, \ldots, x_m, y_1, \ldots, y_n) \, dm(x_1, \ldots, x_m, y_1, \ldots, y_n) \]

In QM, a **Hilbert space** is replaced by a **complex Hilbert space**, and a state is a vector \( \psi \in \mathcal{H} \) that is normalized
\[ \| \psi \|^2 = \langle \psi, \psi \rangle = 1. \]

An observable is a self-adjoint operator on \( \mathcal{H} \) that is densely defined:
\[ \langle \psi, A \phi \rangle = \langle A \psi, \phi \rangle \]

for all \( \phi, \psi \in \mathcal{D}(A) \) (the domain of \( A \)).

The **expected value** of \( A \) in the state \( \psi \):
\[ \text{Ex}_{\psi}(A) = \langle \psi, A \psi \rangle \in \mathbb{R}. \]

**Simple example**: \( \mathcal{H} = \mathbb{C}^2 \), a "qubit", a "spin-1/2" with observable \( A = \mathbb{M}_2(\mathbb{C}) \).

(\( A \) is spanned by a vector space by \( \sigma \) and the Pauli matrices:
\[ \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \].)
Note: $\mathcal{M}_n(C)$ is a C*-algebra:

* algebra: a vector space
  * equipped with a multiplication
  * with a norm
  * with an involution $A \mapsto A^*$ (the adjoint).

C*-property: $\|A^*A\| = \|A\|^2$

In general: the algebra of observables of a quantum system is a C*-algebra.

A state is a positive, normalized functional on $A$.

$\omega : A \rightarrow C$

$A \mapsto \omega(A)$

s.t. $\omega(A^*A) \geq 0$ for all $A \in A$.

$$\sup_{A \in A} \frac{|\omega(A)|}{\|A\|} = 1.$$ (recall $\omega_{\psi} : \psi \mapsto \omega_{\psi}(A^*A) = \langle \psi, A^*A\psi \rangle = \langle A^* A\psi, \psi \rangle = \|A^*A\| \geq 0$

$$\sup_{A \in A} \frac{|\omega_{\psi}(A)|}{\|A\|} = \sup \frac{|\langle A, A\psi \rangle|}{\|A\|} = 1$$)

On $\mathcal{M}_2(C)$: $\omega_{\psi}(\cdot)$ are not all states, hence $\rho = \frac{\omega_{\psi}(\cdot)}{\omega_{\psi}(I)}$.

Let $g \in \mathcal{M}_2(C)$ be not.
1. \[ 0 \leq \mathcal{B} = \mathcal{B}^* \leq M \]
2. \[ \text{Tr}(\mathcal{B}) = 1 \]

Then: \( A \rightarrow \text{Tr}(\mathcal{B}A) \) is a state over \( \mathcal{M}_n(C) \)

"density matrix"

Remark on unbounded observables.
Standard examples such as the position or the momentum of a particle are unbounded. However, if \( A \in \mathcal{A} \) is an element of a C*-algebra, it is necessarily bounded: \( ||A|| < \infty \). Fortunately, we are saved by functional calculus:

\[ \mathcal{U}(t) = e^{itA} \quad (t \in \mathbb{R}) \]
\[ \mathcal{R}_a(t) = (A-t)^{-1} \quad (t \in \text{resolvent set}) \]

are bounded operators.

Composite systems. The state space of a composite system must contain those states that are completely determined by a pair of states of each individual system, as well as linear combinations thereof.

\[ \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \]

vector space tensor product.
For the mathematician: The vector space $\mathcal{H}^{(v)} \otimes \mathcal{H}^{(w)}$ and the bilinear map $t: \mathcal{H}^{(v)} \times \mathcal{H}^{(w)} \to \mathcal{H}^{(v)} \otimes \mathcal{H}^{(w)}$ are uniquely (up to isomorphism) defined by:

For any bilinear form $b: \mathcal{H}^{(v)} \times \mathcal{H}^{(w)} \to \mathbb{C}$, there is a linear form $l: \mathcal{H}^{(v)} \otimes \mathcal{H}^{(w)} \to \mathbb{C}$ such that

$$l \circ t = b$$

i.e. $$l (v^{(v)} \otimes v^{(w)}) = l(v^{(v)}, v^{(w)})$$

i.e.

$$\mathcal{H}^{(v)} \times \mathcal{H}^{(w)} \xrightarrow{t} \mathbb{C}$$

$$\mathcal{H}^{(v)} \otimes \mathcal{H}^{(w)} \xrightarrow{l} \mathbb{C}$$

For the physicist: Pick bases $e^{(v)}_v, e^{(w)}_w$. Then

$$\{ e^{(v)}_v \otimes e^{(w)}_w : 1 \leq v \leq \dim(\mathcal{H}^{(v)}), 1 \leq w \leq \dim(\mathcal{H}^{(w)}) \}$$

is a basis of $\mathcal{H}^{(v)} \otimes \mathcal{H}^{(w)}$.

Scalar product:

$$\langle v^{(v)} \otimes v^{(w)}, w^{(v)} \otimes w^{(w)} \rangle := \langle v^{(v)}, v^{(w)} \rangle_{\mathcal{H}^{(v)}} \cdot \langle w^{(v)}, w^{(w)} \rangle_{\mathcal{H}^{(w)}}$$

Finite-dimensional case: $\dim \mathcal{H}^{(v)} = d_j < \infty$. Then

$$\mathcal{H} = \mathcal{H}^{(v)} \otimes \mathcal{H}^{(w)} = \mathcal{C}^{d_j \times d_k}$$
Algebras \( A = \mathcal{M}_{d_0} \otimes \mathcal{M}_{d_1} = \mathcal{B}(\mathcal{H}) \)  
where \((A \otimes B)_{i,j,k,l} = a_{i-j} b_{k,l}\).

- Finite quantum spin system:
  \(\Lambda\): finite set
  For any \( x \in \Lambda \), we have \( \mathcal{H}_x \cong \mathcal{C}^{d_x}_x \), \( d_x \geq 2 \).
  \[ \mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \cong \mathcal{C}^{\prod_{x \in \Lambda} d_x}_x \]
  and \( A_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) \).

Notes:
- If \( X \subseteq \Lambda \), we can define analogously \( \mathcal{H}_X \), \( A_X \), and \( \mathcal{A}_X \).
  \( \mathcal{A}_X \) is naturally identified with \( A \otimes \mathcal{M}^{\Lambda \setminus X} \in A_\Lambda \),
  i.e. we consider \( A_X \) as a subalgebra of \( A_\Lambda \).
- For now, \( \Lambda \neq \omega \). We will see the subtle changes in the infinite case later.

It will often be important that \( \Lambda \) is equipped with a metric structure \( d : \Lambda \times \Lambda \to [0, \infty) \) satisfying the \( d \)-seq.

There is no usual identification of \( \mathcal{H}_x \) as a subspace of \( \mathcal{H}_\Lambda \).
We will often be interested in infinite systems:

\((\Gamma, d)\): countable metric space (e.g., \(\mathbb{Z}^d\))

(typically a graph with graph distance)

\(x \in \Gamma \Rightarrow H_x = C^d_x\)

For any \(A \subset \Gamma\) finite: \(H_A, \Lambda_A\).

\(A^{loc} : = \bigcup_{x \in A} \Lambda_x\)

Local observables

\(C^\infty\) algebra of quasi-local observables

\(\Lambda_A = \bigcup_{x \in A} \Lambda^{loc}_x\)

Remark: There is no good notion of the Hilbert space of the infinite system (in fact, there are many of them).

However, the notion of states remains meaningful

\[\omega : A \rightarrow \mathbb{C}\]

positive, linear, normalized.
Finally, we will need a way to restrict observables \( A \in \mathcal{A}_n \) to \( \mathcal{A}_n \) where \( \mathcal{A}_n \) is a partial trace.

Let \( D \) be a matrix acting on \( \mathbb{C}^n \otimes \mathbb{C}^n \).

Then
\[
C \mapsto \text{Tr}(D (A \otimes C))
\]
defines a linear functional on \( \mathcal{M}_n(\mathbb{C}) \).

Then by the Gelfand representation (with \( \mathcal{M}_n(\mathbb{C}) \) seen as a Hilbert space with \( \langle X, Y \rangle = \text{Tr}(X^* Y) \)), there is a unique element \( \text{Tr}_n(D) \in \mathcal{M}_n(\mathbb{C}) \) such that
\[
\text{Tr}(D (A \otimes C)) = \text{Tr}(\text{Tr}_n(D) C)
\]
for all \( C \in \mathcal{M}_n(\mathbb{C}) \).

\( \text{Tr}_n(D) \) is called the partial trace of \( D \).

Of course, there is also \( \text{Tr}(D) \in \mathcal{M}_n(\mathbb{C}) \):
\[
\text{Tr}(D (B \otimes N)) = \text{Tr}(\text{Tr}_n(D) B).
\]
Dynamics and propagation estimates

- Time evolution in Schrödinger's ON:

\[ \Psi \rightarrow U(t)\Psi \quad , \quad t \in \mathbb{R} \]

s.t. \[ \| \Psi \| = \| U(t)\Psi \| \quad \text{for all } t \in \mathbb{R} \]

+ \[ t \mapsto U(t)\Psi \] is continuous
+ \[ U(t)U(s)\Psi = U(t+s)\Psi \]

i.e. \[ \{ U(t) : t \in \mathbb{R} \} \] is a strongly continuous group of unitaries on \( \mathcal{H} \).

Stone's theorem: There is a one-to-one correspondence between these and self-adjoint operators \( \mathcal{H} \):

\[ U(t) = \exp(-it\mathcal{H}), \quad \mathcal{H} = \mathcal{H}^* \quad (x) \]

So in order to characterize the time evolution of a quantum system, it suffices to give its Hamiltonian \( \mathcal{H} \).

Physically: \[ \langle \Psi , \mathcal{H}\Psi \rangle \] is the energy of the system in the state \( \Psi \).

(\( \Psi \)) can be written equivalently \( \Psi \): For \( \Psi(t) = U(t)\Psi \):

\[ \begin{cases} \frac{d}{dt} \Psi(t) = \mathcal{H}\Psi(t) \quad \text{"Schrödinger's equation"} \\ \Psi(0) = \Psi \end{cases} \quad , \quad \Psi \in \mathcal{D}(\mathcal{H}) \]

or \[ \frac{1}{i} \frac{d}{dt} \Psi(t) = \mathcal{H}\Psi(t), \quad \Psi(0) = \Psi \]
The evolution of expectation values,

\[ t \mapsto \langle U(t) \hat{A} U(t)^\dagger \rangle = \langle \hat{A} \rangle, \quad (U(t)^\dagger A U(t))_t \]

is "Heisenberg picture".

Dynamics of observables: \[ t \mapsto T_t(A) = U(t)^\dagger A U(t) \]

With \( t \mapsto T_t(A) \) continuous group of \( \mathbb{U} \) - automorphisms.

1. \[ T_{t+s}(A) = U(t+s)^\dagger A U(t+s) = U(t+s)^\dagger U(s)^\dagger A U(t+s) U(s) = T_t(T_s(A)) = (T_t \circ T_s)(A) \]

2. \[ \| T_t(A) \| = \| U(t)^\dagger A U(t) \| = \| A \| \]

3. \[ \| T_{t+\varepsilon}(A) - T_t(A) \| = \| T_\varepsilon(A) - A \| = \| U(\varepsilon)^\dagger A U(\varepsilon) - A \| \]

4. \[ \leq \int_0^\varepsilon \| - i U(s)^\dagger [H, A] U(s) \| ds \]

\[ \leq C_{\varepsilon}[A] \| A \| \quad \text{wherever} \quad A \in \mathcal{D}(\mathcal{H}, \mathcal{A}) \]

5. \[ T_t(AB) = U(t)^\dagger AB U(t) = U(t)^\dagger A U(t) U(t)^\dagger B U(t) = T_t(A) T_t(B) \]

6. \[ T_t(A^k) = U(t)^\dagger A^k U(t) = (U(t)^\dagger A U(t))^k = (T_t(A))^k \]
On a general \( C^* \)-algebra, the dynamics is formulated to be given by a strongly continuous group of \( * \)-automorphisms.

Let's construct a Hamiltonian for a d-particle system. Example: Heisenberg model on a finite lattice \( \Lambda \mathbb{Z} \):

\[
H_n = \sum_{(x,y) \in \Lambda \times \Lambda} \sum_{k=1}^d J_k(x,y) \sigma_x^{k-1} \tau_y^k
\]

\( \sigma_x^k \) is Pauli matrix at site \( x \).

The interaction between sites \( x \) and \( y \).

Typically, \( J_k(x,y) = 0 \) if \( d(x,y) > 1 \).

Also, there could be more than two-body interaction such as

\[
\sigma_x^{k_1} \sigma_x^{k_2} - \sigma_y^{k_1} \sigma_y^{k_2}
\]

If \( J_k(x,y) \) is uniformly integrable in \( y \),

then

\[
|H_n| \leq C |N| \sup_{x \in \Lambda} \sum_{y \in \Lambda} |J_k(x,y)|
\]

\( \leq C' |N| \quad \text{uniformly in } \Lambda \)

no limit (in here) of \( \|H_n\| \) as \( \Lambda \to \Gamma \)
In general, an interaction is a map

\[ \phi : \mathcal{F}(\Gamma) \to \mathcal{A}_\Gamma^1 \]

\[ \chi \mapsto \overline{\phi}(\chi) = \phi(\chi) \] \[ \in \mathcal{A}_x \]

describes the interaction between sites within \( x \).

For any \( A \in \mathcal{F}(\Gamma) \):

\[ H_A = \bigcup_{x \in A} \overline{\phi}(x) \]

and for any \( A \subseteq \mathcal{A}_n \):

\[ \tau^A_t(A) = e^{\int_A \phi} A \]

is a strongly continuous group of \( \mathcal{A} \)-automorphisms of \( \mathcal{A}_n \).

Note: \( e^{\phi} \) acts trivially outside \( A \), so that \( \tau^A_t \) can be seen as acting on \( \mathcal{A}_\Gamma^1 \), with \( \mathcal{A}_n \) a invariant subspace.

Question: when does \( \lim_{t \to \infty} \tau_t^A(A) \) make sense?

Answer: yes if \( \|\phi(\chi)\| \) decays sufficiently rapidly in the diameter of \( X \).

How?: Consider a sequence \( \{A_n\}_{n \in \mathbb{N}} \)

\[ \chi \times A_n \in \mathcal{F}(\Gamma) \]

\[ \chi \times A_n \subseteq \chi \times A_m \] \[ n < m \]

\[ \chi \times A_n \subseteq \chi \times A \] \[ \forall n \geq n_0 \]

\[ A \subseteq A_n \] \[ \forall n > n_0. \]
and prove that the sequence 
\[ \{ T_t \}_{t \in \mathbb{N}} \]
is Cauchy, uniformly for \( t \in [0, T] \).

Key tool: a propagation estimate called Lieb-Robinson bound.

Let \( F: [0, \infty) \to (0, \infty) \) be such that

1) \( \| F \| = \sup_{x \in \mathbb{R}^d} F(d(x, y)) < \infty \).

2) \( \exists C_F \text{ s.t. for any } x, y \in \mathbb{R}^d \)
\[ \sum_{\text{ter}} F(d(x, t)) F(d(y, t)) \leq C_F \tag{1} \]

Example: \( \mathbb{G} = \mathbb{Z}^d \), \( F(r) = \frac{1}{(1 + r)^d + c} \) yield \( C_F = 2^{1 + c} \| F \| \).

Remark: If \( F \) satisfies these properties, then so does
\[ F^\mu(r) = e^{-\mu F(r)} \]
will \( \| F^\mu \| < \| F \| \) and \( C_{F^\mu} \leq C_F \).

Now define a norm for interactions
\[ \| \phi \|_F = \sup_{x \in X} \frac{1}{\sum_{x_0 \in X} F(d(x, x_0))} \sum_{x_0 \in X} \| \phi(x) \|_1 \]
and \( B_f(R) = \{ \text{interactions } \phi: \| \phi \|_F < \infty \} \)
is a Banach space.