The mathematical structure of quantum mechanics.

In QM, the state of $N$ particles is characterized by specifying their position and momenta, i.e., it is a point in phase space $\mathbb{R}^{2N}$.

More generally, in the presence of statistical uncertainty, a state is a probability measure $\mu$ on $\mathbb{R}^{2N}$.

A state allows one to compute the expected value of observables, which are real-valued functions $f \in C_0^\infty(\mathbb{R}^{2N})$ (the set of continuous functions vanishing at $\infty$).

E.g., the energy:

$$E(x, p) = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{1 \leq i < j \leq N} \frac{\alpha}{|x_i - x_j|}$$
and given a state $\rho$, 

$$\mu(\rho) = \int \psi \, d\rho,$$

where 

$$\int \mathcal{D}(\underline{x_1}, \ldots, \underline{x_n}, \underline{p_1}, \ldots, \underline{p_n}) \, d\rho(\underline{x_1}, \ldots, \underline{x_n}, \underline{p_1}, \ldots, \underline{p_n}).$$

In QM, a *Hilbert space* is replaced by a complex *Hilbert space*, and a *state* is a vector $\psi \in \mathcal{H}$ that is normalized:

$$\|\psi\|^2 = \langle \psi, \psi \rangle = 1.$$ 

An observable is a self-adjoint operator on $\mathcal{H}$:

$$\langle \psi, A \phi \rangle = \langle A \psi, \phi \rangle$$

for all $\phi, \psi \in \mathcal{D}(A)$ (the domain of $A$).

The expected value of $A$ in the state $\psi$:

$$\text{tr}_\psi(A) = \langle \psi, A \psi \rangle \in \mathbb{R}.$$ 

*Simple example*: $\mathcal{H} = \mathbb{C}^2$, a "qubit", a "spin-$1/2$" with observable $A = M_2(\mathbb{C})$. $A$ is spanned as a vector space by $I$ and the Pauli matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Note: $\mathcal{M}_n(C)$ is a $C^*$-algebra.

- Algebra: a vector space equipped with a multiplication.
- with a norm
- with an involution $A \mapsto A^*$ (the adjoint).
- $C^*$-property: $\|A^*A\| = \|A\|^2$

In general, the algebra of observables of a quantum system is a $C^*$-algebra.

A state is a positive, normalized linear functional on $A$.

$\omega : A \to C$

$A \mapsto \omega(A)$

$s.t. \quad \omega(A^*A) \geq 0 \quad \text{for all } A \in A$

\[
\sup_{A \in A} \frac{\omega(A)}{\|A\|} = 1
\]

(recall $\omega_{\psi} : \psi \mapsto \omega_{\psi}(A^*A) = \langle A^*A\psi, \psi \rangle = \langle AA^*\psi, \psi \rangle = \|A\psi\|^2 \geq 0$

\[
\sup_{A \in A} \frac{\omega_{\psi}(A)}{\|A\|} = \sup_{A \in A} \frac{\|\psi\|^2}{\|A\|} = 1
\]

On $\mathcal{M}_2(C)$, $\omega_{\psi}(\cdot)$ are not all states true for $\psi$.

Let $\phi \in \mathcal{M}_2(C)$ be not.
\[
0 \leq \|A\| \leq \|A^*\| \\
\text{Tr}(A) = 1
\]

Then: \[ A \mapsto \text{Tr}(A) \] is a state over \( M_n(\mathbb{C}) \) 

**Remark on unbounded observables.**

Standard examples such as the position or the momentum of a particle are unbounded. However, if \( A \in A \) is an element of a C*-algebra, it is necessarily bounded \( \|A\| < \infty \). Fortunately, we are saved by functional calculus:

\[
U_A(t) = e^{itA} \quad (t \in \mathbb{N}) \\
R_A(t) = (A-t)^{-1} \quad (t \in \text{resolvent set})
\]

are bounded operators.

**Composite systems.** The state space of a composite system must contain those states that are completely determined by a pair of states of each individual system, as well as linear combinations thereof.

\[
\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}
\]

vector space tensor product.
For the mathematician: The vector space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ and the bilinear map $t : \mathcal{H}^{(1)} \times \mathcal{H}^{(1)} \to \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ are uniquely (up to isomorphism) defined by:

For any bilinear form $b : \mathcal{H}^{(1)} \times \mathcal{H}^{(1)} \to \mathcal{C}$, there is a linear form $l : \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} \to \mathcal{C}$ such

\[ l_{ot} = b \]

i.e.

\[ l(v^{(1)} \otimes v^{(1)}) = l(v^{(1)}, v^{(1)}) \]

i.e.

\[ \mathcal{H}^{(1)} \times \mathcal{H}^{(1)} \xrightarrow{b} \mathcal{C} \]

\[ \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} \xrightarrow{l_{ot}} \mathcal{C} \]

For the physicist, rich basis $\{ e_n^{(1)}, e_m^{(1)} \}$. Then

\[ \{ e_n^{(1)} \otimes e_m^{(1)} : 1 \leq n < \dim(\mathcal{H}^{(1)}) \}
\]

is a basis of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$.

Scalar product:

\[ \langle v^{(1)} \otimes w^{(1)}, w^{(1)} \otimes w^{(1)} \rangle := \langle v^{(1)}, v^{(1)} \rangle_{\mathcal{H}^{(1)}} \cdot \langle w^{(1)}, w^{(1)} \rangle_{\mathcal{H}^{(1)}} \]

Finite-dimensional case: dim $\mathcal{H}^{(1)} = d_j < \infty$

\[ \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} = \mathcal{C}^{d_j \times d_j} \]
Algebras: \( A = M_{d \times d} = M_d \otimes M_d = \mathcal{B}(\mathcal{H}) \)

where \((A \otimes B)_{ij,kl} = a_{ij}b_{kl}\).

- Finite quantum spin system:
  \(\Lambda\): finite set
  For any \(x \in \Lambda\), we have \(\mathcal{H}_x \cong \mathbb{C}^{d_x}\), \(d_x \geq 2\).

  \(\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \cong \mathbb{C}^{\prod_{x \in \Lambda} d_x}\)

  and \(A_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)\).

Notes: If \(X \subset \Lambda\), we can define analogously \(\mathcal{H}_X, A_X\), and \(A \otimes A_X\) is naturally identified with \(A \otimes M_{\Lambda \setminus X} \subset A_\Lambda\)

i.e. we consider \(A_X\) as a subalgebra of \(A_\Lambda\).

\(\bigotimes\) For now, \(|\Lambda| < \infty\). We will see the subtle issues arising in the infinite case later.

\(\otimes\) It will often be important that \(\Lambda\) is equipped with a metric structure \(d: \Lambda \times \Lambda \to [0, \infty)\) satisfying the \(\delta\)-ineq.

\(\otimes\) There is no natural identification of \(\mathcal{H}_X\) as a subspace of \(\mathcal{H}_\Lambda\).
We will often be interested in infinite systems: 

\((\Gamma, \mathcal{d})\): countable metric space (e.g. \(\mathbb{Z}^d\))

typically a graph with graph distance

\[ x \in \Gamma \implies H_x = C^\infty \]

For any \( \Lambda \subset \Gamma \) finite: \( H_\Lambda, A_\Lambda \).

\[ A^\text{loc} = \bigcup_{\Lambda \subset \Gamma} A_\Lambda \]  

"local observables"

\[ C^\infty \text{-algebra of "quasi-local observables"} \]

\[ \gamma[\Lambda] \]  

**Remark:** There is no good notion of the Hilbert space of the infinite system (in fact, there are many of them).

However, the notion of states remains meaningful:

\[ \Phi : A \rightarrow C \]

positive, linear, normalized.
Finally, we will need a way to restrict observables $A \in A_n$ to $A_n'$ where $\lambda' A \lambda$ is partial trace.

Let $D$ be a matrix acting on $C^m \otimes C^n$. Then

$$C \mapsto \text{Tr}(D \otimes C)$$

defines a linear functional on $M_n(C)$.

Then by the right representation (with $M_n(C)$ seen as a Hilbert space with $\langle X, Y \rangle = \text{Tr}(X^* Y)$), there is a unique element $\text{Tr}_n(D) \in M_n(C)$ such that

$$\text{Tr}(D \otimes C) = \text{Tr}((\text{Tr}_n(D) \otimes C)$$

for all $C \in M_n(C)$.

$\text{Tr}_n(D)$ is called the partial trace of $D$.

Of course, there is also $\text{Tr}(D) \in M_m(C)$:

$$\text{Tr}(D \otimes B \otimes \Lambda) = \text{Tr}(\text{Tr}_n(D) \otimes B).$$


Dynamics and propagator estimates

- Time evolution in Schrödinger's ODE

\[ \Psi \rightarrow U(t) \Psi, \ t \in \mathbb{R} \]

\[ \text{s.t. } \| \Psi \| = \| U(t) \Psi \| \text{ for all } t \in \mathbb{R} \]

\[ t \rightarrow U(t) \Psi \text{ is continuous} \]

\[ U(t)U(s)\Psi = U(t+s)\Psi \]

i.e. \[ \{ U(t) : t \in \mathbb{R} \} \text{ is a strongly continuous group of unitaries on } \mathcal{H}. \]

Stone's Theorem: There is a one-to-one correspondence between these and self-adjoint operators on \( \mathcal{H} \):

\[ U(t) = \exp(-itH), \ H = H^* \ (x) \]

In order to characterize the time evolution of a quantum system, it suffices to give its Hamiltonian \( H \).

Physically: \( \langle \Psi, \ H \Psi \rangle \) is the energy of the system in the state \( \Psi \).

(\( H \) can be written equivalently) \( \Psi \): For \( \Psi(t) = U(t)\Psi \):

\[ \begin{cases} \frac{d}{dt} \Psi(t) = H \Psi(t) & \text{"Schrödinger's equation"} \\ \Psi(0) = \Psi \end{cases} \quad \Psi \in \mathcal{D}(\mathcal{H}) \]

or \( \frac{d}{dt} U(t) = H U(t) \), \( U(0) = 1 \).
The evolution of expectation values:

\[ t \rightarrow \langle U(t)^{\dagger} A U(t) \rangle = \langle \psi, (U(t)^{\dagger} A U(t)) \psi \rangle \]

as "Heisenberg picture".

Dynamic of observables: \[ t \rightarrow T_t(A) = U(t)^{\dagger} A U(t) \]

with: \[ t \rightarrow T_t(A) \] continuous group of \[ \Phi \rightarrow \text{automorphism} \]

\[ T_{t+s}(A) = U(t+s)^{\dagger} A U(t+s) = U(t)^{\dagger} U(s)^{\dagger} A U(s) U(t) \]

\[ = T_t \left( T_s \left( A \right) \right) = (T_t \circ T_s)(A) \]

\[ \| T_t(A) \| = \| U(t)^{\dagger} A U(t) \| = \| A \| \]

\[ \| T_{t+\varepsilon}(A) - T_t(A) \| \leq \| T_{\varepsilon}(A) - A \| \]

\[ = \| U(\varepsilon)^{\dagger} A U(\varepsilon) - A \| \]

\[ \leq \int_{0}^{\varepsilon} \| -i U(s)^{\dagger} [H, A] U(s) \| ds \]

\[ = \| [H, A] \| \leq C_A \| A \| \text{ whenever } A \in \mathcal{D}([H, \cdot]) \]

\[ T_t(AB) = U(t)^{\dagger} A B U(t) = U(t)^{\dagger} A U(t) U(t)^{\dagger} B U(t) \]

\[ = T_t(A) T_t(B) \]

\[ T_t(A^k) = U(t)^{\dagger} A^k U(t) = (U(t)^{\dagger} A U(t))^k = (T_t(A))^k \]
On a general $C^*$-algebra, the dynamics is formulated (to be) given by a strongly continuous group of $*$-automorphisms.

Let's construct a Hamiltonian for a 2D example: Heisenberg model on a finite lattice $\Gamma$:

$$H = \sum_{\langle x, y \rangle} J_{x,y}(x,y) \sum_{\Lambda} \frac{1}{2} \left( \sigma_x^{x,y} \sigma_y^{x,y} + \sigma_y^{x,y} \sigma_x^{x,y} + \sigma_z^{x,y} \sigma_z^{x,y} \right)$$

$J_{x,y}(x,y)$ is uniformly integrable in $y$,

Then, $\|H\| \leq C' |N| \sup_{x \in \Lambda} \sum_{y \in \Lambda} |J_{x,y}(x,y)|$

$\leq C' |N|$

uniform in $\Lambda$

no limit (in $\|H\|$) of $\|H\|$ as $\Lambda \to \Gamma$
In general, an interaction is a map

\[ \phi : \mathcal{F}(G) \rightarrow \mathcal{A}_G^{\text{loc}} \]

\[ \chi \mapsto \phi(\chi) = \phi\chi \in \mathcal{A}_X \]

describes the interaction between sites within \( X \).

For any \( \Lambda \in \mathcal{F}(G) \):

\[ H_{\Lambda} = \bigcup_{X \in \Lambda} \phi(\chi) \]

and for any \( A \subset \Lambda \):\[ \tau_t^A(A) = e^{2 \pi i H_{\Lambda} A} e^{-i t \Lambda} \]

is a strongly continuous group of \( \kappa \)-automorphisms of \( \mathcal{A}_X \).

Note: \( e^{i t \Lambda} \) acts trivially outside \( \Lambda \), so that \( \tau_t^A \) can be seen as acting on \( \mathcal{A}_G^{\text{loc}} \), with \( \mathcal{A}_\Lambda \) a invariant subspace.

Question: when does \( \lim_{t \to \infty} \tau_t^A(\Lambda) = 0 \) hold true?

Answer: yes if \( \| \phi(\chi) \| \) decays sufficiently rapidly in the diameter of \( X \).

How? Consider a sequence \( \Lambda_n \) such that

\[ \Lambda_n \subset \Lambda_{n+1} \quad \text{and} \quad \Lambda_n = \bigcup_{X \in \Lambda_n} \phi(\chi) \]

1. If \( X \in \mathcal{F}(G) \), then \( \exists n_0 \) such that \( X \subset \Lambda_n \) for all \( n > n_0 \).
and prove that the sequence 

\[ \{ T_t \} \]

is Cauchy, uniformly for \( t \in [0, T] \).

Key tool: a propagation estimate called Lieb–Robinson bound.

Let \( F : [0, \infty) \to (0, \infty) \) be such that

1. \( \| F \| = \sup_{x \in \mathbb{R}} F(d(\mu_1)) < \infty. \)

2. \( \exists C_F \text{ s.t. } \forall x, y, t \in \mathbb{R} \)

\[ \sum_{t \in \mathbb{Z}} F(d(x, t)) F(d(y, t)) \leq C_F F(d(x, y)) \]

Example: \( \Gamma = \mathbb{Z}^d \) and \( F(r) = \frac{1}{(1+|r|)^{d+c}} \) yield \( C_F = 2^{d+c} \| F \| \).

Remark: If \( F \) satisfies these properties, then so does

\[ F_\mu(r) = e^{-\mu r} F(r) \]

will \( \| F_\mu \| < \| F \| \) and \( C_\mu F \leq C_F \).

Now define a norm for interactions

\[ \| \Phi \|_F = \sup_{x, y, \mu} \frac{1}{X_{C_F, \mu}} \sum_{x_0 \in X, \ell} \| \Phi(\ell) \| \]

and \( \mathcal{B}_F(r) = \{ \text{interactions } \Phi : \| \Phi \|_F < \infty \} \)

is a Banach space.
Remarks: Let $\|\Phi\|_F < \infty$ express the decay of $\|\Phi(X)\|$ in the size of $X$.

If $\Phi \in S_F(G)$, then

$$\|H\| \leq \sum_{x \in \Lambda} \|\Phi(x)\|$$

$$\leq \sum_{\lambda \in G} \sum_{\gamma \in \Lambda} \sum_{x \in \Lambda} \|\Phi(x)\|$$

$$\leq |\Lambda| \sup_{x \in \Lambda} \sum_{\gamma \in \Lambda} f(d(x,\gamma)) \sum_{\xi \in \Lambda} \frac{\|\Phi(x)\|}{f(d(x,\xi))}$$

$$= \|F\| \leq \|\Phi\|_F$$

In other words, $\Phi \in S_F(G)$ implies that $\|H\|$ is proportional to the volume $|\Lambda|$.

Similarly, the total interaction energy per spin is finite:

$$\sup_{\Lambda \in \mathcal{F}(G)} \sum_{x \in \Lambda} \sum_{\gamma \in \Lambda} \|\Phi(x)\| \leq \|F\| \|\Phi\|_F$$

Theorem (Lieb–Robinson, Neveu–Schulz): Let $\Phi \in S_F(G)$, and $X, \gamma, \Lambda \in \mathcal{F}(G)$ with $X \cap \gamma \neq \emptyset$. Let $A \in \mathcal{A}_X$, $B \in \mathcal{A}_\gamma$. Then:
\[ \| [T^t(A), B] \| \leq \frac{2 \| A \| \| B \|}{C_f} (e^{2 \| \Phi \|_F C_f k t} - 1) D(X,Y) \]

for all \( t \in \mathbb{R} \), where

\[ D(X,Y) = \sum_{x \in X \atop y \in Y} F(d(x,y)) \].

**Lemma**

(i) \( D(X,Y) \leq |X| \cdot |Y| \cdot F(d(X,Y)) \)

(ii) \( \| \Phi \|_F \) is exponentially decaying

\( \Phi \in \mathcal{B}_F(\Gamma) \) for \( \mu > 0 \),

\[ D(X,Y) \leq \sum_{x \in X \atop y \in Y} F(d(x,y)) e^{-\mu d(X,Y)} \]

\[ \leq \min \{ |X|, |Y| \} \| F \| e^{-\mu d(X,Y)} \]

in which case:

\[ \| [T^t(A), B] \| \leq C_a \min \{ |X|, |Y| \} e^{-\mu (d(X,Y) - \frac{\| F \|}{C_f} k t)} \]

where \( C_a = \frac{2 \| A \| \| B \| \| F \|}{C_f} \) and \( t \geq 1 \)

\[ v = \frac{2 \| \Phi \|_F C_f}{\mu} \]

is the Lieb-Robinson velocity.
(iii) Why is the LR-bound a propagation estimate?

Corollary: Let \( Ax_t \). Under the assumption 1 (see Remark 2), there exists \( \tilde{A}_t \in A_x \) such that

\[
\| T_t(A) - \tilde{A}_t \| \leq \frac{2\| A \| \| X \|}{C_F} \| F \| e^{-\lambda r}
\]

for any \( r \in \mathbb{N} \).

Here \( X(n) = \{ x \in X : \text{dist}(x, X) \leq n \} \).

Let \( X \) be a bounded space, \( f \) an interval of \( IR \) and \( A : I \to B(X) \) a continuous function. (w.r.t. the operator norm) Consider the ODE:

\[
\begin{align*}
\partial_t S(t) &= A(t) S(t), \\
S(0) &= S_0 \in X.
\end{align*}
\]

Denote the map \( S_0 \to S(t) \) by \( \gamma_t \), namely

\( \gamma_t(S_0) = S(t) \). We say that \( A \) is \underline{norm preserving} if \( \gamma_t \) is an isometry for all \( t \in I \):

\[
\| \gamma_t(S_0) \| = \| S_0 \| \quad \forall t \in I.
\]

Typical example: \( A(t) = iH \) with \( H = H^* \) on \( H \)

\( X = H \) is a Hilbert space

\( \gamma_t(Y_0) = U_t Y_0 \).
Lemma: Let $A(t)$ be as above. For any continuous $B: t \to X$, the solution of

$$
\begin{align*}
\frac{d}{dt} T(t) &= A(t)T(t) + B(t) \\
T(t_0) &= T_0
\end{align*}
$$

satisfies:

$$
\|T(t) - \Psi_{t,t_0}(T_0)\| \leq \int_{t_0}^{t} \|B(s)\| \, ds
$$

Proof. By Peano’s Principle: (choose $t_0 = 0$, $t > 0$).

$$
T(t) = \Psi_t(T_0) + \int_{0}^{t} \Psi_{t,s}(B(s)) \, ds
$$

is a solution of (x).

Hence

$$
\|T(t) - \Psi_t(T_0)\| \leq \int_{0}^{t} \|\Psi_{t,s}(B(s))\| \, ds
$$

$$
= \|B(t)\|.
$$

Proof of the LRB: Let $\gamma(t) = [T_t(A), B]$. Then

$$
\gamma'(t) = \gamma\left[T_t\left([H, A]\right), B\right]
$$

$$
= [H^X, A], \quad \text{where} \quad H^X = \sum_{t \in \mathbb{N} \times \mathbb{R}} \phi(t).
$$
A two-parameter family of unitary operators $U(t,s)$ on a Hilbert space $\mathcal{H}$ such that:

1. $U(t,s)U(s,r) = U(t,r)$
2. $U(t,t) = 1$
3. $U(t,s)$ is strongly continuous in $(t,s)$

is called a propagator.

**Theorem**: Let $\mathcal{H}$ be a strongly continuous and such that $H(t) = H(t)^*$ and $H(t)^2 = H(t)$. Then there is a propagator $s.t$ for all $\psi \in \mathcal{H}$,

$$
\Psi_s(t) = U(t,s)\psi
$$

subject to,

$$
\begin{align*}
\frac{d}{dt}\Psi_s(t) &= H(t)\psi_s \\
\Psi_s(s) &= \psi
\end{align*}
$$

**Proof**: Define

$$
U(t,s)\psi = \psi + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t=n}^{t_1} \cdots \int_{s}^{t_n-1} H(t_1) \cdots H(t_n) \psi \, dt_n \cdots \, dt_1
$$

Note the order:

$$
t_n \leq t_{n-1} \leq \cdots \leq t_s
$$

Since $H(t)$ is uniformly bounded on $[s,t]$, we have that the $n$th term is bounded above by

$$
\frac{(t-s)^n}{n!} \left( \sup_{t \in [t,s]} \|H(t)\| \right)^n \|\psi\|
$$

since $t_n \leq t_{n-1} \leq \cdots \leq t_s$

proving that the series converges in norm.
Now: \( U(t,t)^N = 1 \) is immediate

\( U(s,t)^{-1} = U(t,s)^N \), too

Multiply the series for \( U(t,s) \) and \( U(s,r) \)
and reorganizing the terms yields the cocycle property.

Hence

\[
U(s,t)^N U(s,t) = U(t,s) U(s,t) = U(t,t) = \mathbb{1}
\]

and similarly for \( U(t,s) U(t,t)^N \), proving
uniqueness.

Finally: the series being uniformly convergent, it is
differentiable term by term so that

\[
\frac{d}{dt} U(t,s)^N = -iH(t) U(t,s)^N + \sum_{n=1}^\infty (-i)^{n+1} H(t) \int_t^{s_n} \int_s^{s_{n-1}} \ldots \int_s^{t_n} H(t_{n-1}) \ldots H(t_1) \lambda(t_n) dt_n \ldots dt_1 U(t,s)^N
\]

\[
= -iH(t) U(t,s)^N \quad \text{indeed.} \quad \square
\]

Remarks: It could be shown that \( U(t,s)^N \) is the
unique solution of \((A)\).

If \( A \in B(H) \), this translates to

\[
T(t,s)(A) = U(t,s)^N A U(t,s)
\]

being the unique solution of

\[
i \frac{d}{dt} T(t,s)(A) = T(t,s) \left( [A, H(t)] \right)
\]
as well as

\[ T_{t,s}(A) = U_1(t,s) A U(t,s)^* \]

hence the unique solution of

\[ i \frac{d}{dt} T_{t,s}(A) = \begin{bmatrix} H(t), & T_{t,s}(A) \end{bmatrix} \]

In particular, the map \( i : \mathbb{R} \to [0, T(t)] \) is norm-potential.
\[ \begin{align*}
\Delta_{\text{Koba}} &= -i \left[ [\tau_c(A), B], \tau_c(HX) \right] - i \left[ [B, \tau_c(HX)], \tau_c(A) \right] \\
&= -i \left[ \tau_c(HX), \tau_c(HX) \right] - i \left[ \tau_c(A), [\tau_c(HX), B] \right].
\end{align*} \]

Note that \(-i \left[ \cdot, \tau_c(HX) \right] \) is norm preserving. Hence

\[ \| [\tau_c(A), B] \| \leq \| [A, B] \| + 2 \| A \| \int_0^t \| [\tau_c(HX), B] \| \, ds. \]

We define

\[ M_B(X_1, t) := \sup_{A \in U_B} \| [\tau_c(A), B] \| \]

and obtain

\[ M_B(X_1, t) \leq M_B(X_1, 0) + 2 \int_0^t M_B(2, s) \, ds. \]

with the initial condition \( M_B(2, 0) \leq 2 \| B \| \| \delta_4(2) \|

where \( \delta_4(t) = \begin{cases} 1 & \text{if } \exists \eta \neq \emptyset \ (t = \sup \{ \eta \}) \\ 0 & \text{otherwise} \end{cases} \)

Hence:

\[ M_B(X_1, t) \leq 2 \| B \| \sum_{n=0}^{\infty} \frac{\delta_4(t)}{n!} \]

where \( \delta_4 = \sum_{n=0}^\infty \frac{\delta_4}{n!} \sum_{i=0}^n \prod_{k=1}^i \Phi(t_k) \delta_4(t_{n-k}) \)
\[ a_n = \sum_{\forall x \in X} \| \Phi(x) \| \sum_{\forall y \in Y} \| \Phi(y) \| \]

\[ \leq \sum_{\forall x \in X} \sum_{\forall y \in Y} \sum_{\forall t \in T} \| \Phi(x) \| \sum_{\forall y \in Y} \| \Phi(y) \| \]

\[ \leq \sum_{\forall x \in X} \sum_{\forall y \in Y} F(d(x, t)) \sum_{\forall y \in Y} \| \Phi(y) \| \]

\[ \leq \sum_{\forall x \in X} \sum_{\forall y \in Y} F(d(x, t)) F(d(y, t)) \| \Phi \|_F^2 \]

\[ \leq C_F \| \Phi \|_F \sum_{\forall x \in X} \sum_{\forall y \in Y} \]

\[ \leq C_F \| \Phi \|_F^n C_F^{n-1} D(X, Y) \]
We conclude:

\[ V_{cb} (X_{i}, t) \leq \frac{2 \| \phi \|_{t}}{C_{t}} \sum_{n=0}^{\infty} \frac{(2 \pi \| \phi \|_{t} C_{t})^{n}}{n!} \cdot D(X_{i}, t) \]

and \( e^{-\theta |\Omega_{C_{t}}|} \) from the assumption that \( X_{i} = \emptyset \) so that \( \omega = 0 \).

The next crucial consequence of the LRB theorem: the exponential clustering theorem. If there is a spectral gap \( g \), above the ground state energy, the correlation in a ground state decay exponentially:

\[
\left| \langle \Omega, A \beta \Omega \rangle - \langle \Omega, A \Omega \rangle \langle \Omega, \beta \Omega \rangle \right| \\
\leq C(A, \beta, g) e^{-\mu d(X, i)}
\]

(3)

for \( A \in A_{X}, \beta \in A_{Y}, \) and \( \mu = \frac{\xi \gamma}{g + 4 \| \phi \|_{t}} \).

Notes: the decay rate \( \mu \) is sometimes called "the gap".  
1) \( \Omega \) were a pure bra-ket product \( \Psi \Omega \).  
2) \( \langle \Omega, A \beta \Omega \rangle = \langle \Psi, A \Psi \rangle \langle \Phi, B \Phi \rangle \) and (4) show that a gapped ground state is "close to" being a product.

* in relativistic quantum field theory: a simple proof uses the exact propagation within a cone and \( \mu \) is exactly the gap.
Setting a quantum spin system with dynamics

\[ T_t(A) = e^{itH} A e^{-itH} \]

with \( H \geq 0 \), \( \ker(H) \neq \emptyset \), and spectral gap \( g > 0 \)

\[ g = \sup \{ \delta > 0 : \sigma(H) \cap (0, \delta) = \emptyset \} \]

Remark: This could be in finite volume, or in the GNS representation of the infinite volume algebra/dynamics.

1. \( H \geq 0 \) should be understood as \( H \) is bounded below, \( g \) is a choice of normalization.

Denote \( P_0 \) the projection onto \( \ker(H) \).

Theorem: Let \( a > 0 \) and \( \phi \in \mathcal{S}_a \) (exponentially decay interaction). Let \( \Omega = P_0 \Omega \) and \( \| \Omega \| = 1 \).

There exists \( \mu > 0 \) s.t. \( \| A e^{A_x} \psi \|_1 \) with \( d(x, y) > 0 \) and \( P_0 B \psi = P_0 B^* \psi = 0 \), the bound

\[ \langle \Omega, A e^{bX} \psi \rangle_1 \leq C(A, B, g) e^{-\mu d(X, Y)} \left( 1 + \frac{g^2}{\mu^2 d(X, Y)^2} \right) \]

holds for all \( 0 \leq \mu g \leq 2 \mu d(X, Y) \).

Concretely:
\[ \mu = \frac{2g}{\nu + 4L \parallel B \parallel C_2} \]

\[ C(A, B, g) = \|A\| B \parallel \left( 1 + \frac{1}{\sqrt{\nu d(x, y)}} \right) + \frac{2 \parallel F \parallel}{\nu C_2} \min \left\{ \|X\|, \|Y\| \right\} \]

* Note: Let \( R_{\text{lin}}(P_0) = 1 \) for \( B = B^* \), the theorem can be applied to \( \vec{B} = B - \langle \Omega, B \Omega \rangle \cdot \Omega \), where \( \text{Re} \Omega = \Omega \). At \( b = 0 \), we recover (4), p. 19.

* Elements of proof: Define \( j : \mathbb{R} \rightarrow \mathbb{C} \) by

\[ j(t) = \langle \Omega, e^{itB} \Omega \rangle \]

By the spectral theorem,

\[ j(t) = \langle \Omega, A^* \Omega \rangle + \int_0^1 e^{itE} \, d \langle \Omega, A^* \Omega \rangle \]

\[ = 0 \]

by assumption.

Let \( b > 0 \), \( T > 5 \). Then

\[ j(t) = \frac{1}{2\pi i} \int_{\gamma - ib} \frac{j(t)}{t - IT} \, dt \]

with

\[ |j(e^{iT})| \leq \|A\| \|B\| e^{-T \sin \Theta} \]

for any \( \Theta \in [0, \pi] \).
Hence, the integral over the semi-circle vanishes as \( T \to \infty \):

\[
| f(t) | \leq \lim_{T \to \infty} \left| \frac{1}{2\pi i} \int_{-i}^{i} \frac{1}{t - ib} \, dt \right|
\]

Let \( a > 0 \) (to be chosen later) and

\[
f(t) = e^{-ab^2} \left( \frac{1}{2\pi} \int_{-i}^{i} e^{-at \zeta} \, d\zeta + \frac{1}{2\pi} \int_{-i}^{i} (e^{-a\zeta^2} - e^{-at\zeta}) \, d\zeta \right)
\]

Lemma: \( E \in \mathbb{R}, \ b \in (0, \infty) \):

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{-i}^{i} \frac{e^{iEt} e^{-at\zeta}}{t - ib} \, dt = \frac{1}{2\pi} \int_{0}^{\infty} e^{-lw} e^{-\frac{(lw + E)^2}{4a}} \, dl.
\]

Since we are interested in taking \( b \to 0^+ \), we will bound the L.H.S. by the pure Gaussian.

Now \( E > g > 0 \) and the integral \( \int e^{-\frac{lw}{2}} \, dl \) is finite.

But we write:

\[
\langle \Omega, A t E(b/\Omega) \rangle = \langle \Omega, t\zeta(b)\Omega \rangle + \langle \Omega, [A, t\zeta(b)] \Omega \rangle
\]

Now, the first term has the correct sign:

\[
\langle \Omega, t\zeta(b)\Omega \rangle = \int_{0}^{\infty} e^{-iEt} \langle \Omega, B \, d\zeta(t)\Omega \rangle
\]

(Here we use that \( P_0 B^2 L = 0 \)).

and this first contribution is bounded above by

\[
C ||A|| ||B|| e^{-\frac{g}{2a}}.
\]
The second term is bounded by
\[ \frac{1}{b_0} \int_{t_0}^{t_0} \left[ A \cdot \zeta(t) \right] e^{-\alpha t} dt. \]

Use the LRB! \textbf{Truncate at} \( T_0 \):
\[ \int_{T_0}^{T_0} (\ldots) < C e^{-\alpha (d(X_i, Y) - v T_0)} \quad \text{(LRB)} \]
\[ \int_{T_0}^{T_0} (\ldots) < C e^{-v T_0^2} \quad \text{(Gaussian)} \]
which decay is \( d(X_i, Y) \), by picking \( T_0 \) to be a function of \( d(X, Y) \).

The remaining term can be bounded by a result similar to the lemma. The claim follows by optimizing over \( \alpha, T_0 \) (independently of \( b \)).

So far, we have considered Q.S.P. The results hold equally well for lattice fermions (with few threats).

1. Hilbert space of a particle hopping on \( \Lambda \): \( h_{\Lambda} = \ell^2(\Lambda) \)
2. \( \mathcal{H}^N_{\Lambda} = \bigotimes_{j=1}^{N} h_{\Lambda} \)
3. **Note**: \( \mathcal{H}^N_{\Lambda} = \ell^2(\Lambda \times \Lambda \times \cdots \times \Lambda) \)
4. Action of \( S_\pi \): \( \mathcal{P}^N_{\pi} : \Psi(x_{\pi(1)}, \ldots, x_{\pi(N)}) \rightarrow \Psi(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(N)}) \)
5. Define \( \mathcal{H}^N_{\Lambda, F} = \{ \Psi \in \mathcal{H}^N_{\Lambda} : \mathcal{P}^N_{\pi} \Psi = (-1)^{\text{sgn}(\pi)} \Psi \} \)
   the "fermionic" (antisymmetric subspace of \( \mathcal{H}^N_{\Lambda} \)
6. \textbf{Finally} \( \mathcal{H}_0^N = h_{\Lambda} \); \( \mathcal{H}_0 = \mathbb{C} \), and the...
Termionic Fock space

$$\mathcal{F}_\Lambda = \bigoplus_{j=0}^{\infty} \mathcal{H}_{\Lambda,F}^N$$

(Note that the sum is truncated at $N=\Lambda F$.)

Interpretation: $\psi = (\psi_0, \psi_1, \psi_2, \ldots)$, $\psi_j \in \mathcal{H}_{\Lambda,F}^j$

where $|\psi_0|^2$ is the probability to have no particle

$$\langle \psi_0 | \psi_0 \rangle \qquad 1 \text{ particle}$$

$|\psi_j|^2$ is the probability to

Have these $j$ particles at sites $x_1, \ldots, x_j$.

For $f \in \mathcal{H}_\Lambda$, let $b(f): \mathcal{H}_\Lambda^N \to \mathcal{H}_\Lambda^{N-1}$

$$b(f)(g_{e_1} \cdots g_{e_N}) = \frac{1}{\sqrt{N}} \langle f, g_{e_1} \rangle \cdots \langle f, g_{e_N} \rangle$$

Fact: $b(f) p_{ji} = p_{ji} b(f)$

Hence we define $b(f)$ on $\mathcal{F}_{\Lambda,F}$ by

Adding the condition $b(f) \mathcal{H}_\Lambda^0 = 0$.

"Annihilation operator"$

Let $b(f)^*$ be its adjoint, $\mathcal{H}_\Lambda^{N-1} \to \mathcal{H}_\Lambda^N$

$$b(f)^* \phi = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (\text{sgn}(-1)^{j-1} p_{ji}(f \otimes \phi))$$

where $p_{ji}^{-1} = (h, i, h, \ldots, h, h, i, \ldots, h, h, i, \ldots, h)$
Clearly:  $H_{\lambda} = \text{span} \{ b(\lambda) \varepsilon_\lambda : \varepsilon_\lambda \in h_{\lambda} \}$

where is the vector $\{ 1, 0, 0 \ldots \}$

"the vacuum.

Similarly:  $H_{\lambda}^* = \text{span} \{ b(\lambda) - b(\lambda) \varepsilon_\lambda : \varepsilon_\lambda \in h_{\lambda} \}$

$|b(\lambda)|$: creation operator

$b(\lambda)$: annihilation operator

A calculation yields the canonical anticommutation relations:

\[
\begin{align*}
|b(\lambda)| (|b(\gamma)| + |b(\lambda)| |b(\gamma)|) &= \langle \varepsilon_\gamma, \varepsilon_\lambda \rangle \\
|b(\lambda)| (|b(\gamma)|^* |b(\lambda)|) &= 0 \\
|b(\lambda)| (|b(\gamma)|^* |b(\lambda)|^*) &= 0
\end{align*}
\]

Notation:  $b_x = b(\delta x)$,  $b_x^* = b(\delta x)^*$,  $x \in \Lambda$

"annihilate and create a particle at $x$" and note:  $b_x^* b_x = 0$

"Pauli's exclusion principle"

(more generally: there cannot be two fermions in the same state:  $(b(\lambda))^2 = 0$)

The C*-algebra generated by $\{1, b(\lambda), b(\lambda)^* : \varepsilon_\lambda \in h_{\lambda} \}$
is denoted $\text{CAR}(h_{\lambda})$. 
So far so good. We can further define the algebra in the infinite volume limit
\[ \tilde{A}_\Gamma = \text{CAR}(\ell^1(\Gamma)) \]
which is generated by \( b_x, \tilde{b}_x, \forall x \in \Gamma \) satisfying the canonical anticommutation relation.

Notes: \( A_{\Gamma} \) may be realized on Fock space \( \mathcal{F}_\Gamma \)
but it does not need to be.

Since \( \ell^1(\Gamma) \) is naturally a subspace of \( \ell^1(\Gamma) \)
\( \mathcal{A}_{\Gamma} \) is naturally embedded in \( \Omega(\ell^1(\Gamma)) \).

Locally: for \( f, g \in \ell^1(\Gamma) \) will
\[ \text{supp}(f) \cap \text{supp}(g) = \emptyset \]
\( b(f) \) anticommute with \( b(g) \).

- Number operator: \( x \) at site \( x \):
  \[ b_x^\dagger b_x \]
  Indeed
  \[ \langle \psi, b_x^\dagger b_x \psi \rangle = \langle \psi, \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \rangle \]
  \[ = |\mathcal{F}|^2 \]
  is the probability to find the particle at \( x \).

And \( N_x = \sum_{x \in \Lambda} b_x^\dagger b_x \)

\( \forall A \in \tilde{A}_\Gamma \) and \( \forall \lambda \in \Lambda \) we define the parity automorphism
\[ \theta^\Lambda(A) = e^{-i \pi N_\Lambda} A e^{i \pi N_\Lambda} \]
which extend uniquely to an automorphism of $\tilde{A}_r$.

First of all: $\Theta^2 = \text{id}$, because $\sigma(N_h) \subset N$ implies $\exp(2\pi i N_h) = \text{id}$.

Hence $\sigma(\Theta^2) = [-1, 1]$.

and we let $A^\pm_r$ to be the corresponding eigenspace.

"even/odd" elements, $A^+_r$ is a $C^*$ subalgebra.

Lemma: Let $X, Y \in \tilde{A}_r$ st $X_n Y = 0$. Then

(i) for any $A \in A^+_r$, $B \in A^-_r$

\[ [A, B] = 0 \]

(ii) if $A \in A^-_r$, $B \in A^+_r$ are such that $[A, B] = 0$, then either $A \in A^+_r$ or $B \in A^-_r$.

Proof: See Exercises.

A moral of the story: results from 2.5.5 hold for fermions under the condition that all actors are even: The "test observables", the interaction term $\Phi(X)$.

Typical interaction: $H = \sum \left( h(x, y) a_x^\dagger a_y + \overline{h(x, y)} a_y^\dagger a_x \right) \quad \text{"happy"} \quad + \sum_{x} v(x) n_x \quad \text{"external potential"}$

$+ \sum_{x, y \in A} w(x, y) n_x n_y \quad \text{"interaction"}$.
Remark: if \( h \) is a finite dimensional Hilbert space, \( \text{CA}_h(h) \) is isomorphic with the C*-algebra \( M_{2n}(\mathbb{C}) \), where \( n = \dim(h) \).

Examples: \( n = 1 \)

\[
a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

\[
a^+a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\( \text{Span} \ M_{2n}(\mathbb{C}) \), and indeed \((a^+) = (a)^2 = 0\).

In general: Jordan-Wigner transformation.

Pick a basis \( \{ \psi_1, \ldots, \psi_n \} \) of \( h \), and let

\[
e_k^{(\pm)} = \psi_k \pm \psi_k^\dagger \quad e_k^{(i)} = \psi_n \quad e_k^{(\dagger)} = \psi_n^\dagger
\]

where \( \psi_n = \frac{1}{\sqrt{M}} \left( M - 2 \Delta \psi_j^\dagger \psi_j \right) \).

Check: \( \{ e_{ij}^{(j)} \} \) are \( n \) families of mutually commuting \( 2 \times 2 \) matrix units.

\[
e_{ij} e_{kl} = \delta_{jk} e_{il}
\]

\[
[e_k^{(\pm)}, e_l^{(\dagger)}] = 0 \quad (k \neq l)
\]

and the map is invertible:

\[
b(\psi_k) = \left( \prod_{j=1}^{k-1} (e_{11}^{(j)} - e_{12}^{(j)}) \right) e_{12}^{(k)}
\]
Rotations & Spins

Given \( \Phi \in \mathcal{H} \), the corresponding state is
\[
\omega_\Phi \Phi = \frac{\langle \Phi, \Phi \rangle}{\| \Phi \|^2}
\]
in particular, \( \omega_\Phi \) can be identified with the projector \( \mathcal{P}_\Phi \) of \( \Phi \), or equivalently the one-dimensional projector
\[
\mathcal{P}_\Phi \Phi = \frac{\langle \Phi, \Phi \rangle}{\| \Phi \|^2} \Phi
\]

Define \( \bar{\Pi}(\mathcal{H}) = \{ \mathcal{P}_\Phi : \Phi \in \mathcal{H} \} \).

Def.: A symmetry is a map \( S: \bar{\Pi}(\mathcal{H}) \to \bar{\Pi}(\mathcal{H}) \) s.t.
\[
\text{Tr}(\mathcal{P}_\Phi \mathcal{P}_\Psi) = \text{Tr}(S(\mathcal{P}_\Phi)S(\mathcal{P}_\Psi))
\]
equivalently
\[
|\langle \Phi, \Psi \rangle|^2 = |\langle S(\Phi), S(\Psi) \rangle|^2
\]

Theorem (Wigner): Every invertible symmetry is represented by
\[
S(\mathcal{P}) = U \mathcal{P} U^\dagger
\]
where \( U \) is either a unitary or an antiunitary operator.

\( U \) is unique up to a phase.

Remark: Here, we consider only the unitary case.

Given a Hamiltonian on \( \mathcal{H} \), \( S \) is called a symmetry
The dynamics of
\[
U e^{-\mathcal{T} t} = e^{-\mathcal{T} t} U
\]

naturally
\[
[U, \mathcal{H}] = 0
\]

Many symmetries come in the guise of a Lie group $G$.

A projective representation of $G$ on $\mathcal{H}$ is an invertible symmetry \( \{ S_g : g \in G \} \), such that
\[
S_g \circ S_h = S_{gh} \quad (g, h \in G)
\]
and $g \mapsto S_g(\mathcal{H})$ is continuous.

Since Wigner's theorem determines $U$ only up to a phase
\[
U_g U_h = \omega(g, h) U_{gh} \quad , \quad \omega(g, h) \in U(1)
\]
and
\[
\omega(g, gh) = \omega(g, h)
\]

We will be interested in $G = SO(3)$ or $G = SU(2)$.

We start by considering unitary representations of $SO(3)$.

$U : SO(3) \to B(\mathcal{H})$

$R \mapsto U(R)$

5. $U(R_r R_c) = U(R_c) U(R_r), \quad U(-) = \overline{U}$

6. $U(id) = 1$

7. $(U(R))^* = U(R^*)$
Simple example: \( \mathcal{H} = L^2(\mathbb{R}^3) \) 

\[
(\mathcal{U}_t(\mathbb{R}^2))(x) = \Psi(R^t x)
\]

- Lie algebra \( \mathfrak{so}(3) \). Consider a differentiable curve \( t \mapsto R(t) \) s.t. \( R(0) = \text{id} \). If tangent vector at \( \text{id} \):

\[
\Omega = \frac{d}{dt} R(t) \bigg|_{t=0}
\]

\( \Omega \) they form a Lie algebra with bracket given by the commutator:

\[
x \times \Omega_1, + x \times \Omega_2 = \frac{d}{dt} \left( R_a(x, t) R_b(x, t) \right) \bigg|_{t=0}.
\]

\[
\Omega \times \Omega^{-1} = \frac{d}{dt} \left( R_a(x, t) R_b(x, t) \right) \bigg|_{t=0}.
\]

and hence:

\[
[\Omega_1, \Omega_2] = \frac{d}{dt} \left( R_a(t) R_b R_a(t)^{-1} \right) \bigg|_{t=0}
\]

are all tangent vectors.

Now:

\[
R(t)^T R(t) = I = \sigma_t^T + \Omega = 0
\]

and

\[
\sigma_t^T + \Omega = 0 \implies (e^{\sigma t})^T (e^{\sigma t}) = e^{(\sigma^T + \Omega) t} = I
\]

Hence:

\( \mathfrak{so}(3) \) is exactly the set of antisymmetric elements of \( \mathfrak{u}_3(\mathbb{R}) \):

\[
\mathfrak{u}_3(\mathbb{R}) = \begin{pmatrix}
0 & -V_3 & V_2 \\
V_3 & 0 & -V_1 \\
-V_2 & V_1 & 0
\end{pmatrix}
\]

It follows that \( \dim_{\mathbb{R}}(\mathfrak{so}(3)) = 3 \) with basis

\[
\{ e_i : i = 1, 2, 3 \}.
\]

Note: \( e^{\sigma t} \) is a rotation of angle \( \theta \) around \( \frac{v}{||v||} \).
Commutation relation:
\[[\Omega, \Omega] = \Omega_3 \quad (+ \text{cyclic})\]

Indeed: A calculation shows that
\[R_\Omega(u)R_\Omega^{-1} = \Omega(R_\Omega(u)) \quad (R \in SO(3))\]
whose infinitesimal version yields
\[[\Omega(v_1), \Omega(v_2)] = \Omega(v_1 \wedge v_2)\]

We turn to representation of \(SO(3)\). If \(U\) is an arbitrary
representation of \(SO(3)\), then
\[U(\Omega) = \frac{d}{dt} U(R(t)) \bigg|_{t=0}\]
is a representation of the algebra \(\mathfrak{so}(3)\).

\[U(\alpha \Omega, \beta \Omega, + \alpha \Omega, \beta \Omega) = \frac{d}{dt} U(R_{\alpha}(x, t) R_{\beta}(x, t)) \bigg|_{t=0}\]
\[= \frac{d}{dt} U(R_{\alpha}(x, t)) U(R_{\beta}(x, t)) \bigg|_{t=0} = \alpha \Omega U(\Omega) + \beta \Omega U(\Omega)\]

Similarly \(U([\alpha \Omega, \beta \Omega]) = [U(\Omega), U(\Omega)]\)

Furthermore: \(U(R(t)) U(R(t)^*) = I \rightarrow U(\Omega) + U(\Omega)^* = 0\).

In physics: the preferred operator are
\[\Pi(v) = i U(\Omega(v)) \quad \text{will}\]
\[\Pi(v)^* = \Pi(v)\]
\[[\Pi_1, \Pi_2] = i \Pi_3 \quad (+ \text{cyclic}) \quad \text{"angular momentum operators"} \]
We are trying to classify irreducible representation of \( \mathfrak{so}(3) \) : reps. s.t. \( \mathfrak{h} \) are the only invariant subspace.

1. Algebraic relation: Let

\[ \Pi^+ = \Pi_x + i \Pi_y \]

Then:
\[ [\Pi_3, \Pi^\pm] = \pm \Pi^\pm \] (1)
\[ [\Pi_3, \Pi_3] = 2 \Pi_3 \] (2)

Let further
\[ \Pi^2 = \Pi_x^2 + \Pi_y^2 + \Pi_3^2 \]

Then:
\[ [\Pi_3, \Pi_j] = 0 \quad (j=1, 2, 3) \] (3)
\[ \Pi_j^2 = \Pi^2 + \Pi_3 (\Pi_3 \pm 1) \] (4)
and finally:
\[ \Pi_j = \Pi_j \quad (j=1, 2, 3) \]
\[ \Pi^\pm = \Pi^\mp \] (5)

2. Spectral analysis: Let \( \lambda \in \mathfrak{h}(\Pi^2) \)

By (3), its eigenspace is invariant under \( \Pi_j \)

\[ \Pi_j = \lambda \Pi_j \]

By irreducibility, it is all of \( \mathfrak{h} \), hence
\[ \Pi^2 = \lambda \Pi \] (6)

Let now \( \psi \in \sigma(\Pi_3) \) and
\[ \Pi_3 \Psi = \lambda \Psi \]
By (1):
\[ \Pi_3 \Pi_+ \Psi = \Pi_+ (\Pi_3 \pm \Pi) \Psi = (\mu \pm 1) \Pi_+ \Psi \tag{7} \]
so that \( \mu \pm 1 \in \sigma(\Pi_3) \) provided \( \Pi_+ \Psi \neq 0 \).

Moreover, (4, 6) imply:
\[ \Pi_+ \Pi_+ \Psi = (\lambda - \mu (\mu \pm 1)) \Psi \tag{8} \]

The space of all these eigenvectors (for all possible values of \( \mu \)) is invariant. Hence it is all of \( \mathcal{H} \).

But \( \langle \Psi, \Pi_+ \Pi_+ \Psi \rangle = ||\Pi_+ \Psi||^2 \geq 0 \) (by (5)),

and (8) imply that there can be only a finite number of them.

\[
= n \quad \text{dim } \mathcal{H} < \infty.
\]

In particular: \( \exists j \in \mathbb{N} \) and \( \Psi_j, \|\Psi_j\| = 1, \forall j \).

\[ \Pi_3 \Psi_j = j \Psi_j \quad \text{and} \quad \Pi_+ \Psi_j = 0. \]

and (8) yields:
\[ \lambda \Psi_j = j (j+1) \Psi_j \]

since \( \Pi_3 \Pi_+ \Psi_j = 0 \),

\[ \lambda = j (j+1) \]
Define recursively
\[ c_m \Psi_{m-1} = \Pi \Psi_m \quad (m = j, j-1, \ldots) \]
will \( c_m > 0 \) s.t.
\[ \| \Psi_{m-1} \| = 1. \]
By the above
\[ \Pi \Psi_m = c_m \Psi_m \quad (m = j, j-1, \ldots) \]
The recursion stops when
\[ c_m^2 = \| \Pi \Psi_m \|^2 = \langle \Psi, \Pi^2 \Psi \rangle = 0, \]
namely when
\[ j(j+1) - m(m-1) = 0 \]
that is:
\[ m = -j \]
(only solution since \( m \leq j \))

Conclusion: There are \((2j+1)\) eigenvalues of \( \Pi \)
\[ \{-j, -j+1, -j+2, \ldots, j-1, j\} \]
Since \( 2j+1 \in \mathbb{N} \), we have
\[ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \]
as the only possibilities

Theorem: The irreducible unitary representations of \( so(3) \), denoted \( D_j \) are parametrized by non-negative half-integers with:
\[ \dim D_j = 2j+1 \]
and
\[ \Pi^2 = j(j+1) \Pi. \]
Remarks: We have constructed the basis:
\[ \{ \psi_{j,m} \mid m = -j, \ldots, j \} \] st.
\[ \Pi_3 \psi_{j,m} = m \psi_{j,m} \]
\[ \Pi^2 \psi_{j,m} = j(j+1) \psi_{j,m} \]
\[ \Pi^2 \psi_{j,m} = \sqrt{j(j+1) - m(m+1)} \psi_{j,m} \]

The 2-dim representation \( D_j \) in the standard basis:
\[ \Pi_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Pi_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Pi_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

\( D_j \) is called the "spin-\( j \)" representation.

If the irreps arise from one of \( SO(3) \), then
\[ U(R(\theta e_3)) = e^{-i \theta \Pi_3} \]
and since \( U(R(2\pi e_3)) = I \), we conclude that
\[ \sigma(\Pi_3) \subset \mathbb{Z} \]

Thus \( D_j \) arising from \( SO(3) \) have integer \( j \).

What about the group generated by the half-integer \( D_j \)?

- Group \( SU(2) \): \( U^* U = U \), det \( U = 1 \)
- Algebra \( su(2) \): \( A^* + A = 0 \), \( \text{Tr}(A) = 0 \)

L. Lie algebra with bracket \( [A_i, A_j] = A_i A_j - A_j A_i \)
\[ A e su(2) \text{ is parametrized by } a \in \mathbb{R}^3 \]
\[ A(a) = -\frac{i}{2} \left( \begin{array}{cc} a_3 & a_1 - i a_2 \\ a_1 + i a_2 & -a_3 \end{array} \right) = -\frac{i}{2} \sigma \cdot a \]

Pauli matrices

Check:
\[ [A(\sigma), A(\lambda)] = A(2\sigma \lambda) \quad (9) \]

\[ \dim_{\mathbb{R}}(su(2)) = 3 \text{ with basis } \{ A_i = A(\sigma_i) : i = 1, 2, 3 \} \text{ s.t.} \]
\[ [A_i, A_j] = A_k \quad (+ \text{ cyclic}) \]

Hence \( so(3) \) and \( su(2) \) are isomorphic, with the map given explicitly by (9):
\[ A(\sigma) \rightarrow S(\lambda) \]

\[ \text{since } A \sigma B = S(\lambda) \]

\[ \text{Exponentially (9): } V = e^{A(\sigma) t}, \quad R = e^{S(\lambda) t} \]

\[ V A(\lambda) V^{-1} = A(R \lambda) \quad (10) \]

defines a group homomorphism \( \Phi : SU(2) \rightarrow SO(3) \)
\[ V \mapsto \Phi(V) = R \]

which is not injective: \( \Phi(V) = \Phi(-V) \) \( \text{by (10)} \)

(no other phase is possible since \( \det(e^{iU}) = e^{2i\pi} = 1 \))

Hence \( SO(3) = SU(2) / \{ \pm 1 \} \).
Explicitly, since \((e \cdot e)^2 = 1\) whenever \(\| e \| = 1\),
\[ V(\gamma e) = e^{A(\gamma e)} = e^{-\frac{i}{2} (\gamma \cdot e) \mathbf{H}} \]
\[ = \cos \left( \frac{\gamma}{2} \right) \mathbf{H} - i (\gamma \cdot e) \sin \left( \frac{\gamma}{2} \right) \]

Given a rotation of angle \(\theta\) around \(e\), this formula gives \(V \in SU(2)\) s.t. \(\Phi(V) = \mathbf{R}\).

We see explicitly that \(V(2\pi e) = -\mathbf{I}\), namely \(\Phi(-V) = \mathbf{R}\).

Moreover, \(\mathbf{D} = \frac{\mathbf{H}}{\mathbf{2}}\) provide projective representations of \(\text{SO}(3)\) — which are acceptable actions in quantum mechanics.

Finally, the spin of the electron:

The Hamiltonian of a electron in the presence of a nucleus in robbia invariant \(H_0\) acts on \(L^2(\mathbb{R}^3)\), which carries the representation \(U_0(\mathbb{R})\).

By robbia invariance, eigenspaces of \(H_0\) are invariant under \(U_0\), and hence carry a representation of \(\text{SO}(3)\).

By placing the "spin" in a magnetic field in the direction \(e_3\), the robbia invariance is broken and the degeneracy is (generically) destroyed.
Experimental fact (Zeeman effect):

\[ 2j + 1 = 2N \]

(no magnetic field) (with magnetic field)

Even degeneracy, namely \( j \) is half-integer (electron \( \frac{1}{2} \))

Hence:

The electron carries an "internal" degree of freedom: spin \( \frac{1}{2} \)

\[ H = L^2(M^2) \otimes C \]

carries a representation of \( SU(2) \):

\[ U(V) = U_0(L(V)) \otimes V \]