Solution 5

Problem 1. (i) [Duhamel’s formulae] Let $\mathbb{R} \ni s \mapsto H_s$ be a matrix-valued function that is twice continuously differentiable. Prove that
\[
\frac{d}{ds} e^{H_s} = \int_0^1 e^{\mu H_s} \dot{H}_s e^{(1-\mu)H_s} d\mu
\]
and that if $\tau^s_t(A) = e^{itH_s}A e^{-itH_s}$, then
\[
\frac{d}{ds} \tau^s_t(A) = i \int_0^t [\tau^s_\mu(\dot{H}_s), \tau^s_t(A)] d\mu.
\]
The derivatives above are all considered in the norm topology.

(ii) [Index of projections] Let $\mathcal{H}$ be a separable Hilbert space (not necessarily finite dimensional). Let $P, Q$ be two orthogonal projections such that $P - Q$ is a trace-class operator. Prove that for any $k \in \mathbb{N}$,
\[
\text{Tr}((P - Q)^{2k+1}) \in \mathbb{Z}.
\]
Hint. The operators
\[
C = P - Q, \quad S = P - (1 - Q)
\]
satisfy $CS + SC = 0$ and that $C^2 + S^2 = I$.

Solution. (i) We note that
\[
e^{H_{s+\epsilon}} - e^{H_s} = e^{\mu H_{s+\epsilon}}e^{(1-\mu)H_s}\bigg|_{\mu=0}^{\mu=1} = \int_0^1 \frac{d}{d\mu} e^{\mu H_{s+\epsilon}}e^{(1-\mu)H_s} d\mu = \int_0^1 e^{\mu H_{s+\epsilon}}(H_{s+\epsilon} - H_s)e^{(1-\mu)H_s} d\mu
\]
Hence Duhamel’s formula holds by the continuity of $s \mapsto e^{H_s}$ and differentiability of $s \mapsto H_s$.

We now apply (i) to the two exponentials of $\tau^s_t$ to get
\[
\frac{d}{ds} \tau^s_t(A) = \int_0^1 \left( e^{it\mu H_s} (it) \dot{H}_s e^{it(1-\mu)H_s} A e^{-itH_s} + e^{itH_s} A e^{-it\mu H_s} (-it) \dot{H}_s e^{-it(1-\mu)H_s} \right) d\mu = i \int_0^t [\tau^s_\mu(\dot{H}_s), \tau^s_t(A)] d\mu
\]
since $e^{it(1-\mu)H_s} = e^{itH_s}e^{-it\mu H_s}$ and we substituted $\mu \to 1 - \mu$ in the second term. The claim follows by the substitution of $t\mu \to \mu$.

(ii) The algebraic relations for $C, S$ follow by a direct calculation. Moreover, both $C$ and $S$ are self-adjoint, and $C$ is a compact operator by assumption. Let $\lambda$ be an eigenvalue of $C$ with normalized eigenvector $\psi$. Since $CS + SC = 0$, we conclude that $-\lambda$ is also an eigenvalue of $C$, provided $S\psi \neq 0$. That is the case if and only if $\lambda = \pm 1$ since
\[
0 = \langle S\psi, S\psi \rangle = \langle \psi, (\mathbb{I} - C^2)\psi \rangle = 1 - \lambda^2.
\]
Let now \( \text{Spec}(C) \setminus \{0\} = \{\lambda_j : j \in \mathbb{N}\} \). By Lidskii’s theorem,

\[
\text{Tr}((P - Q)^{2k+1}) = \text{Tr}(C^{2k+1}) = \sum_{j \in \mathbb{N}} \lambda_j^{2k+1} = m_1 - m_{-1}
\]

where \( m_{\pm 1} \) denotes the geometric multiplicity of the eigenvalue \( \pm 1 \). In other words,

\[
\text{Tr}((P - Q)^{2k+1}) = \dim \text{Ker}(P - Q - 1) - \dim \text{Ker}(P - Q + 1) \in \mathbb{Z}.
\]
Problem 2. Consider a quantum spin system on $\mathbb{Z}^d$ equipped with an $F$-function. Let $a > 0$, $\Phi \in \mathcal{B}_{F_a}$, $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$ and let $v_a$ be the corresponding Lieb-Robinson velocity. Let $Z \subset \Lambda$ and $A \in \mathcal{A}_Z$.

(i) Prove that

$$\| [H_\Lambda, A] \| \leq 2 \| A \| \| Z \| \| F_a \|_1 \| \Phi \|_{F_a}.$$  

(ii) Let $\Psi \in \mathcal{B}_{F_a}$ and $K_\Lambda = \sum_{X \subset \Lambda} \Psi(X)$. Prove that if $t \geq 1/v_a$, then

$$\| [K_\Lambda, e^{itH_\Lambda} A e^{-itH_\Lambda}] \| \leq C \| \Psi \|_{F_a} \| Z \|_2 (v_a t)^d,$$

where $C$ is independent of $\Lambda$.

Solution. (i) Since operators with disjoint supports commute, the only interaction terms of $H$ that appear in the commutator are those with supports $X$ such that $X \cap Z \neq \emptyset$. Hence,

$$\| [H, A] \| \leq \sum_{X \cap Z \neq \emptyset} 2 \| \Phi(X) \| \| A \| \sum_{x \in Z} \sum_{y \in \Lambda \setminus X} \text{supp}(\Phi(x,y)) \frac{\| \Phi(X) \|}{F_a(d(x,y))} \leq 2 \| A \| \| \Phi \|_{F_a} \sum_{x \in Z} \sum_{y \in \mathbb{Z}^d} F_a(d(x,y)) \leq 2 \| A \| \| \Phi \|_{F_a} \sum_{x \in Z} F_a(d(x,y)) \leq 2 \| A \| \| \Phi \|_{F_a} \sum_{x \in Z} F_a(d(x,y))$$

uniformly in $\Lambda$.

(ii) For any $Z \subset \Lambda$, we denote $\mathbb{E}_Z(A) = \dim(\mathcal{H}_{\Lambda \setminus Z})^{-1} \text{Tr}_{\mathcal{H}_{\Lambda \setminus Z}} A \in \mathcal{A}_Z$, see Problem 3, Sheet 2. We define $Z^n = \{ x \in \mathbb{Z}^d : \text{dist}(x,Z) \leq v_a t + n \}$, where $v_a$ is the Lieb-Robinson velocity. Then

$$\tau_t(A) = \mathbb{E}_{Z^1}(\tau_t(A)) + \sum_{n=2}^{\infty} (\mathbb{E}_{Z^n}(\tau_t(A)) - \mathbb{E}_{Z^{n-1}}(\tau_t(A)))$$

which is a finite telescopic sum since $\Lambda$ is finite. Denoting $A^1(t) = \mathbb{E}_{Z^1}(\tau_t(A))$ and for $n \geq 2$, $A^n(t) = \mathbb{E}_{Z^n}(\tau_t(A)) - \mathbb{E}_{Z^{n-1}}(\tau_t(A))$, we note that $\text{supp}(A^n(t)) = Z^n$ by construction. Hence, by (i),

$$\| [K_\Lambda, A^n(t)] \| \leq 2 \| A^n(t) \| \| \Psi \|_{F_a} \| Z^n \| \| F_a \|,$$

and we note that

$$|Z^n| \leq \kappa |Z| (v_a t + n)^d \leq \kappa |Z| (2 v_a t)^d$$

whenever $t \geq 1/v_a$. By the Lieb-Robinson bound,

$$\| A^n(t) \| \leq \| \tau_t(A) - \mathbb{E}_{Z^n}(\tau_t(A)) \| + \| \tau_t(A) - \mathbb{E}_{Z^{n-1}}(\tau_t(A)) \| \leq \frac{2 \| A \| \| Z \| \| F \| e^{-a(n-1)} (1 + e^{-a})}{C a}$$

for all $n \geq 2$. Moreover, $\| A^1(t) \| \leq \| A \|$. Altogether, we obtain

$$\| [K_\Lambda, \tau_t(A)] \| \leq 2 \| A \| \| Z \| (v_a t)^d \kappa 2^d \| F \| \| \Psi \|_{F_a} \left( \frac{1}{|Z| \| F \|} + \frac{4}{Ca} \sum_{n=1}^{\infty} n^d e^{-a(n-1)} \right),$$

where the first term is the contribution of $A^1(t)$. This concludes the argument since the series is summable.
**Problem 3.** On $\mathcal{H} = L^2(\mathbb{R}^2; \mathbb{C})$, consider the Landau Hamiltonian

$$H = \frac{1}{2}(P^2 + X^2) - L_3,$$

where

$$(P_1\psi)(x) = -i\partial_1\psi(x), \quad (X_1\psi)(x) = x_1\psi(x), \quad L_3 = X_1P_2 - X_2P_1.$$ 

All operators are defined on Schwarz space $\mathcal{S}$.

Define $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$, as well as $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$.

(i) Prove that $\langle \psi, \partial_z\phi \rangle = -\langle \partial_z\psi, \phi \rangle$ for all $\phi, \psi \in \mathcal{S}$.

(ii) Let $a = \frac{1}{2}Z + \partial_z$. Prove that

$$H = 2a^*a + \mathbb{I}.$$ 

(iii) Prove that $a, a^*$ satisfy the canonical commutation relations:

$$[a, a^*] = \mathbb{I}, \quad [a, a] = [a^*, a^*] = 0,$$

(iv) Prove that for any holomorphic function $\varphi$,

$$\psi_0(z) = \varphi(z)e^{-\frac{|z|^2}{2}}$$

is a ground state of $H$.

(v) Prove that $\{\psi_{0,m}\}_{m \in \mathbb{N}}$ given by

$$\psi_{0,m}(z) = \frac{1}{\sqrt{\pi m!}}z^m e^{-\frac{|z|^2}{2}}$$

is an orthonormal basis of the ground state space (called the lowest Landau level).

(vi) Prove that $\text{Spec}(H) = 2\mathbb{N} + 1$ and construct an orthonormal eigenbasis $\{\psi_{k,m}\}_{(k,m) \in \mathbb{N} \times \mathbb{N}}$ of $H$.

**Solution.** (i) Note first that $\mathcal{S}$ is an invariant subspace for $\partial_z, \partial_{\bar{z}}$. We have

$$\langle \psi, \partial_z\phi \rangle = \int \overline{\psi(x)}(\partial_1 - i\partial_2)\phi(x)dx = \int (-\partial_1 - i\partial_2)\overline{\psi(x)}\phi(x)dx = -\langle \partial_z\psi, \phi \rangle$$

where the second equality is by integration by parts and the fact that $\phi, \psi \in \mathcal{S}$.

(ii) We first observe that

$$X^2 = ZZ = \bar{Z}Z, \quad P^2 = -\partial_1^2 - \partial_2^2 = -4\partial_z\partial_{\bar{z}} = -4\partial_z\partial_z,$$

and that

$$L_3 = -i\left(\frac{1}{2}(Z + \bar{Z})(\partial_z - \partial_{\bar{z}}) - \frac{1}{2i}(Z - \bar{Z})(\partial_z + \partial_{\bar{z}})\right) = Z\partial_z - \bar{Z}\partial_{\bar{z}}.$$ 

Furthermore, for any $\psi \in \mathcal{S}$,

$$\begin{align*}
(\partial_zZ)\psi(x) &= \frac{1}{2}(\partial_1 - i\partial_2)(x_1 + ix_2)\psi(x) = \frac{1}{2}(1 + 1)\psi(x) + (x_1 + ix_2)\partial_z\psi(x) = (\mathbb{I} + Z\partial_z)\psi(x), \quad (1) \\
(\partial_{\bar{z}}Z)\psi(x) &= \frac{1}{2}(\partial_1 + i\partial_2)(x_1 + ix_2)\psi(x) = \frac{1}{2}(1 - 1)\psi(x) + (x_1 + ix_2)\partial_{\bar{z}}\psi(x) = Z\partial_{\bar{z}}\psi(x) \quad (2)
\end{align*}$$
Hence,
\[ H = -2\partial_z\partial_{\bar{z}} + \frac{1}{2}Z\bar{Z} - (\partial_z Z - 1) + Z\partial_{\bar{z}} = 2\left(\frac{1}{2}Z - \partial_z\right)(\frac{1}{2}Z + \partial_{\bar{z}}) + I \]

It remains to note that (i) implies that \( a^* = \frac{1}{2}Z - \partial_z \) (here \( a : \mathcal{S} \to \mathcal{S} \)).

(iii) We write (1) as \([Z,\partial_z] = -I\) which, together with the adjoint relation \([\bar{Z},\partial_{\bar{z}}] = -I\) yields
\[ [a, a^*] = \frac{1}{4} [Z, \bar{Z}] - \frac{1}{2} [Z, \partial_z] + \frac{1}{2} [\partial_z, \bar{Z}] - [\partial_z, \partial_{\bar{z}}] = I. \]

The other commutation relations are derived similarly using (2), namely \([Z, \partial_{\bar{z}}] = 0\),
\[ [a, a] = \frac{1}{4} [Z, \bar{Z}] - \frac{1}{2} [Z, \partial_z] + \frac{1}{2} [\partial_z, \bar{Z}] - [\partial_z, \partial_{\bar{z}}] = 0. \]

(iv) Since \( a^*a \geq 0 \), a ground state is a solution of
\[ (a^*a)\psi_0 = 0. \]

The equation \( a\psi = 0 \) reads explicitly
\[ \partial_z\psi_0(z, \bar{z}) = -\frac{1}{2}z\psi_0(z, \bar{z}) \]
and any function of the form
\[ \psi_0(z, \bar{z}) = \varphi(z, \bar{z})e^{-\frac{1}{2}z\bar{z}} \]
is a solution, provided \( \partial_z\varphi(z, \bar{z}) = 0 \), namely \( \varphi(z, \bar{z}) = \varphi(z) \) is holomorphic.

(v) Polynomials are dense in the set of holomorphic functions, so that a basis of the lowest Landau level is given by any basis of polynomials. We check that the proposed set is an orthonormal basis:
\[ \langle \psi_{0,m}, \psi_{0,l} \rangle = \frac{1}{\pi \sqrt{m!!}} \int_0^{2\pi} e^{-i(m-l)\theta} d\theta \int_0^\infty r^{m+l}e^{-r^2}rdr = 2\pi \delta_{m,l} \frac{1}{\pi m!} \frac{1}{2}\Gamma(m+1) = \delta_{m,l}. \]

where we wrote \( z = re^{i\theta} \).

(vi) We first note that \( (a^*a)a^* = a^*(a^*a) + a^* \), so that for any ground state \( \psi_{0,m} \), the vector \( a^*\psi_{0,m} \) is an eigenvector of \( a^*a \), resp. \( H \), for the eigenvalue 1, resp. 3. Reciprocally, \( (a^*a)a = a(a^*) - a \) implies that any eigenvector \( \psi_1 \) of \( a^*a \) for the eigenvalue 1 is such that \( a\psi_1 \) is a ground state. It follows from (v) that the set \( \{a^*\psi_{0,m} : m \in \mathbb{N}\} \) is an orthonormal basis of \( H \) for the eigenvalue 3. The claim follows similarly by setting
\[ \psi_{k,m} = \frac{1}{\sqrt{k!}}(a^*)^k\psi_{0,m} \]
and noting that
\[ (a^*a)\psi_{k,m} = k\psi_{k,m} \]
since \( (a^*a)(a^*)^k = k(a^*)^{k-1} + (a^*)^k(a^*a) \), namely \( H\psi_{k,m} = (2k + 1)\psi_{k,m} \). It remains to show that there is no other eigenvalue of \( a^*a \). If \( \lambda \in \text{Spec}(a^*a) \) with eigenvector \( \varphi \), then \( (a^*a)\psi = \lambda\psi \) proving that \( \lambda - 1 \in \text{Spec}(a^*a) \), provided \( a\psi \neq 0 \), namely \( \psi \) is not a ground state. Repeating this argument, \( \lambda - m \in \text{Spec}(a^*a) \) for all \( m \) such that \( a^m\psi \neq 0 \). Since \( a^*a \geq 0 \), we conclude that \( \lambda \in \mathbb{N} \) and that \( a^\lambda\psi \) is a ground state. It remains to observe that \( \{\psi_{k,m} : k, m \in \mathbb{N}\} \) is a basis of \( L^2(\mathbb{R}^2; \mathbb{C}) \).
On the density of states. Finally, we compute

\[ n_k(z) = \sum_{m=0}^{\infty} |\psi_{k,m}(z)|^2 \]

which is the number of states per unit area at the point \( z \) in the \( k \)th Landau level. By translation invariance, it suffices to compute it at \( z = 0 \). For \( k = 0 \), the sum reduces to its first term, namely

\[ n_0 = \frac{1}{\pi} \]

For \( k \geq 1 \), we first note that

\[ L_3 \psi_{0,m}(z) = m\psi_{0,m}(z) \]

and since \([L_3, a^*] = -a^*\) (by a long calculation), we further conclude that

\[ L_3 \psi_{k,m}(z) = \frac{1}{\sqrt{k!}} L_3 (a^*)^k \psi_{0,m}(z) = (m - k)\psi_{k,m}(z). \]

Since rotation invariance implies \( L_3 \psi_{k,m}(0) = 0 \), we conclude that \( \psi_{k,m}(0) \neq 0 \) if and only if \( m - k = 0 \) and hence \( n_k = |\psi_{k,k}(0)|^2 \). Writing

\[ a^* = -\frac{|z|^2}{\pi} \partial_z e^{-\frac{|z|^2}{2}}, \]

we compute

\[ \psi_{k,k}(0) = \frac{1}{\sqrt{k!}} (a^*)^k \psi_{0,k}(z)|_{z=0} = \frac{1}{\sqrt{\pi} k!} \frac{|z|^2}{2} (-\partial_z)^k z^k e^{-|z|^2}|_{z=0} = (-1)^k \frac{1}{\sqrt{\pi}} \]

and hence \( n_k = 1/\pi \), too. Restoring the units, we obtain

\[ n_k = \frac{qB}{2\pi c} \]

states by unit area in each Landau level.