Problem 1. Let $P, Q$ be two projections on $\mathcal{H}$. Let
\[
R = (P - Q)^2, \quad \tilde{U} = QP + (1 - Q)(1 - P), \quad \tilde{V} = PQ + (1 - P)(1 - Q),
\]
and
\[
U = \tilde{U}(1 - R)^{-1/2}, \quad V = \tilde{V}(1 - R)^{-1/2},
\]
whenever these operators are well-defined.

(i) Prove that $R$ commutes with both $P$ and $Q$.
(ii) Prove that $[\tilde{U}, \tilde{V}] = 0$ and that $\tilde{U}\tilde{V} = 1 - R$.

From now on, assume that $P, Q$ are orthogonal projections and that $\|R\| < 1$.

(iii) Prove that $U, V$ are unitary with $V^* = U$.
(iv) Prove that
\[
Q = UPU^*.
\]

(v) Let $\mathcal{D}$ be a connected region of $\mathbb{C}$, and let $\mathcal{D} \ni z \mapsto P(z)$ be a continuous function such that $P(z) = P(z)^2 = P(z)^*$ for all $z \in \mathcal{D}$. Prove that $\text{Rank}(P(z))$ is constant.

Solution. (i) Note that
\[
QR = QP - PQP - Q + Q = Q - QPQ \tag{1}
\]
and similarly for $PR = P - PQP$.

(ii) Replacing $Q$ by $\mathbb{I} - Q$ (which is a projection, too) above, we have that
\[
(\mathbb{I} - Q)(\mathbb{I} - P - Q)^2 = (\mathbb{I} - Q) - (\mathbb{I} - Q)P(\mathbb{I} - Q).
\]

Moreover,
\[
R + (\mathbb{I} - P - Q)^2 = \mathbb{I}.
\]

Together,
\[
\mathbb{I} - R = (\mathbb{I} - Q) - (\mathbb{I} - Q)P(\mathbb{I} - Q) + Q(\mathbb{I} - R)
\]
\[
= (\mathbb{I} - Q)(\mathbb{I} - P)(\mathbb{I} - Q) + PQP = \tilde{U}\tilde{V}, \tag{2}
\]
where we used (1) in the form $Q(\mathbb{I} - R) = PQP$. Since $\tilde{U}, \tilde{V}$ are related by the exchange of $P, Q$, the explicit symmetry of last expression implies their commutativity.

(iii) First of all, $\mathbb{I} - R = (\mathbb{I} - P - Q)^*(\mathbb{I} - P - Q)$ so that it is a non-negative self-adjoint operator. It is invertible with positive inverse if $\|R\| < 1$. Hence $(\mathbb{I} - R)^{-1/2}$ is well-defined. But then
\[
UV = \tilde{U}(1 - R)^{-1/2}\tilde{V}(1 - R)^{-1/2}.
\]

By (i), $R$ commutes with both $P, Q$ and hence also with $\tilde{U}, \tilde{V}$. Therefore,
\[
UV = \tilde{U}\tilde{V}(1 - R)^{-1} = \mathbb{I}
\]
by (2). Similarly, $VU = I$ and hence $V^{-1} = U$. Finally, $V^* = \tilde{U}$ if $P, Q$ are orthogonal projections and hence

$$V^* = (1 - R)^{-1/2} \tilde{V}^* = \tilde{U} (1 - R)^{-1/2} = U.$$  

(iv) By definition

$$\tilde{U} P \mathcal{H} \subset Q \mathcal{H}, \quad \tilde{U} (I - P) \mathcal{H} \subset (I - Q) \mathcal{H}, \quad \tilde{V} Q \mathcal{H} \subset P \mathcal{H}, \quad \tilde{V} (I - Q) \mathcal{H} \subset (I - P) \mathcal{H}.$$  

Since $I - R$ commutes with $\tilde{U}, \tilde{V}$ the operators $U, V$ have the same mapping properties. By (iii), they are however invertible so that $=$ instead of $\subset$ hold for $U, V$ instead of $\tilde{U}, \tilde{V}$ above. In turn, these are equivalent to

$$Q = U P U^{-1} = U P U^*, \quad P = U^{-1} Q U = U^* Q U.$$  

(v) Let $t_0$ be fixed and let $0 < \epsilon < 1$. By continuity, there is $\delta > 0$ such that

$$|t - t_0| < \delta \iff \|P(t) - P(t_0)\| < \epsilon,$$

and hence $\|(P(t) - P(t_0))^2\| \leq \|P(t) - P(t_0)\|^2 < 1$. It follows from the above that there is a unitary $U(t, t_0)$ such that

$$P(t) = U(t, t_0) P(t_0) U(t, t_0)^*.$$  

In particular they have the same rank. We conclude with a standard argument. Let $S = \{z \in \mathcal{D} : \text{Rank}(P(z)) = \text{Rank}(P(z_0))\}$ for some fixed $z_0 \in \mathcal{D}$. By the above, $S$ is relatively open. By the same argument, $\mathcal{D} \setminus S$ is also relatively open. Since $S$ is not empty and $\mathcal{D}$ is connected, we must have $S = \mathcal{D}$.  

**Problem 2.** We say that $M(z)$ is an analytic function (in a neighbourhood of $z = 0$) if the Taylor series $M(z) = \sum_{j=0}^{\infty} z^n M_n$ is norm-convergent for all $|z| < \rho$ for some $\rho > 0$. We denote

$$M = M(0), \quad N(z) = M(z) - M.$$ 

Let

$$R(\zeta, z) = (M(z) - \zeta)^{-1}.$$ 

(i) Assume that $\zeta_0$ is not in the spectrum of $M$. Prove that if $|\zeta - \zeta_0|$ is sufficiently small,

$$R(\zeta, z) = R(\zeta_0)(1 - (\zeta - \zeta_0 - N(z))R(\zeta_0))^{-1}.$$ 

(ii) Prove that $z \mapsto R(\zeta, z)$ is analytic, namely

$$R(\zeta, z) = R(\zeta) + \sum_{n=1}^{\infty} z^n R_n(\zeta)$$

for $|z|$ sufficiently small, and give a formula for the coefficients $R_n(\zeta)$.

(iii) Let $\lambda$ be an eigenvalue of $M$, and let $\Gamma$ be a positively oriented circle in $\mathbb{C}$ that encloses $\lambda$ and no other eigenvalue. Prove that if $|z|$ is sufficiently small, then $M(z)$ has no eigenvalue on $\Gamma$.

(iv) Prove that if $|z|$ is sufficiently small the projection

$$P(z) = -\frac{1}{2\pi i} \int_{\Gamma} R(\zeta, z) d\zeta$$

is well-defined and analytic.

(v) Conclude that if $|z|$ is sufficiently small, then $\text{Rank}(P(z))$ is constant.

**Solution.** (i) Simple algebra gives

$$M(z) - \zeta = (M - \zeta_0) - (\zeta - \zeta_0) + N(z) = (I - ((\zeta - \zeta_0) - N(z))R(\zeta_0))(M - \zeta_0).$$

By assumption, $M - \zeta_0$ is invertible with bounded inverse $R(\zeta_0)$. Moreover, if $\zeta$ is so that

$$|\zeta - \zeta_0| < \|R(\zeta_0)\|^{-1} - \|N(z)\|$$

then $I - ((\zeta - \zeta_0) + N(z))R(\zeta_0)$ is invertible (with a convergent Neumann series), from which we conclude that $M(z) - \zeta$ is invertible with

$$R(\zeta, z) = R(\zeta_0)(I - ((\zeta - \zeta_0) - N(z))R(\zeta_0))^{-1}.$$ 

(ii) We set $\zeta = \zeta_0$ in the formula just proven and the Neumann series again to obtain a convergent power series

$$R(\zeta, z) = R(\zeta)(I + N(z)R(\zeta))^{-1} = R(\zeta)\sum_{n=0}^{\infty} (-N(z)R(\zeta))^n = R(\zeta) + \sum_{n=1}^{\infty} z^n R_n(\zeta), \quad (3)$$

where we set

$$R_n(\zeta) := (-1)^n \sum_{n_1, \ldots, n_k \in \mathbb{N} : n_1 + \cdots + n_k = n} R(\zeta)M_{n_1}R(\zeta) \cdots R(\zeta)M_{n_k}R(\zeta).$$
(iii) By assumption, \( R(\zeta) \) has no singularity (and it is uniformly bounded) on \( \Gamma \). But then, the series (3) is convergent for small \(|z|\), uniformly in \( \zeta \) on \( \Gamma \). The existence of \( R(\zeta, z) \), the resolvent of \( M(z) \), for all \( \zeta \) on \( \Gamma \) proves that there is no eigenvalue on \( \Gamma \) for \( z \) in the disc of convergence of (3).

(iv) It is well-defined by the above, since \( R(\zeta, z) \) is uniformly bounded on \( \Gamma \). Since the series (3) converges uniformly, \( P(z) \) has the following convergent power series representation

\[
P(z) = P + \sum_{n=1}^{\infty} z^n \int_{\Gamma} R_n(\zeta) d\zeta,
\]

hence it is analytic.

(v) Immediate corollary of (iv) and Problem 1(v).
**Problem 3.** The Hamiltonian for a spin-$\frac{1}{2}$ in a magnetic field $-B \in \mathbb{R}^3$ is given by

$$H_B = B \cdot \sigma,$$

where $\sigma$ is the vector of Pauli matrices.

(i) Compute the eigenvalues $\lambda_B^\pm$ (where $\lambda_B^- \leq \lambda_B^+$) and normalized eigenvector $\Omega_B^-$ of $H_B$. Adjust the phase so that the first component is real.

(ii) Assume that $B \neq 0$. Compute Berry’s connection

$$A = \langle \Omega_B^-, d\Omega_B^- \rangle,$$

in the coordinates $(B_1, B_2, B_3)$, where $d$ is the exterior derivative:

$$df = \sum_{j=1}^3 \frac{\partial f}{\partial B_i} dB_i.$$

Let now $\|B\| = h$ be fixed. The parameter manifold is $\mathcal{M} = S^2$.

(iii) Express $\Omega_B^-$ in polar coordinates. Characterize the domain $\mathcal{C} \subset S^2$ of this representation.

(iv) Consider now $\Omega_B^- = e^{-i\phi} \Omega_B^-$ and characterize the domain $\hat{\mathcal{C}} \subset S^2$ of this representation. Compute the corresponding $A, \hat{A}$ in polar coordinates and prove that

$$dA = d\hat{A}, \quad \text{on } \mathcal{C} \cap \hat{\mathcal{C}}.$$

(v) Find a function $t : S^1 \rightarrow U(1)$ such that

$$A = t^{-1} \hat{A}t + t^{-1}dt.$$

(vi) Show by an explicit calculation that the total flux is quantized:

$$\frac{1}{2\pi i} \int_{S^2} dA = -1.$$

**Solution.** (i) First of all,

$$\sigma \cdot B = \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix},$$

whose eigenvalues are easily calculated to be $\lambda_B^\pm = \pm\|B\|$. For the eigenvector $\Omega_B^-$, we pick $(\Omega_B^-)_2 = B_1 + iB_2$ to get $(\Omega_B^-)_1 = (\|B\|^2 - B_3^2)/(\|B\| + B_3) = \|B\| - B_3$, and normalized

$$\Omega_B^- = \frac{1}{\sqrt{2\|B\|(\|B\| - B_3)}} \begin{pmatrix} \|B\| - B_3 \\ B_1 + iB_2 \end{pmatrix}.$$

Note that this expression is ill-defined at $B_3 = \|B\|$, see (iii,iv).

(ii) Denoting $\Omega_B^- = \|v\|^{-1}v$, we compute

$$A = \frac{1}{\|v\|^2} \langle v, dv \rangle - \frac{1}{2\|v\|^2} (\langle v, dv \rangle + \langle dv, v \rangle) = \frac{\text{Im} \langle v, dv \rangle}{\|v\|^2}.$$
In this case, the same calculation can be done without an application of Stokes, since orientation, and used (5).

Let \( N \), \( S \) denote the North and South poles of \( \mathcal{M} \). Then \( \lim_{B \to \mathcal{N}} \Omega_B^- = \lim_{(\theta,\phi) \to (0,\phi)} \Omega_{(\theta,\phi)}^- = (0, e^{i\phi}) \) while \( \lim_{B \to \mathcal{S}} \Omega_B^- = \lim_{(\theta,\phi) \to (\pi,\phi)} \Omega_{(\theta,\phi)}^- = (1, 0) \). In other words, \( \lim_{B \to \mathcal{N}} \Omega_B^- \) does not exist, while \( \lim_{B \to \mathcal{N}} \Omega_B^\circ \) is perfectly well defined. Hence \( (\theta, \phi) \mapsto \Omega_{(\theta,\phi)}^- \) is a bijection on \( (0, \pi) \times [0, 2\pi] \), covering \( S^2 \setminus \{N\} \). Note that this reflects the singularity of (4) at \( B_3 = \|B\| \).

(iii) We set \( B_1 = h \sin \theta \cos \phi \), \( B_2 = h \sin \theta \sin \phi \), \( B_3 = h \cos \theta \), and hence

\[
\Omega_{(\theta,\phi)}^- = \begin{pmatrix} \sin(\theta/2) \\ e^{i\phi} \cos(\theta/2) \end{pmatrix}.
\]

Let \( S_1, S_2 \) denote the northern, resp. southern, hemisphere. By Stokes’ theorem,

\[
\Phi = \frac{1}{2\pi i} \left( \int_{S_1^2} d\tilde{A} + \int_{S_2^2} d\tilde{A} \right) = \frac{1}{2\pi i} \left( \int_{S_1} \tilde{A} - \int_{S_1} A \right) = \frac{1}{2\pi i} \int_{S_1} t^{-1}dt = -\frac{1}{2\pi} \int_0^{2\pi} d\phi = -1,
\]

where we noted that the equator is the boundary of both hemispheres, but with an opposite orientation, and used (5).

In this case, the same calculation can be done without an application of Stokes, since

\[
dA = d\tilde{A} = \frac{1}{2i} \sin \theta \, d\theta \wedge d\phi
\]

and

\[
\int_{S_1^2} d\tilde{A} = \frac{1}{2} \int_0^{\pi/2} \int_0^{2\pi} \text{curl} \left( -i(1 - \cos \theta) \right) d\phi d\theta = -i \frac{1}{2} \int_0^{2\pi} d\phi = -\pi i.
\]