Problem 1. Let $A$ be a $C^*$-algebra and let $\omega$ be a state over $A$.

(i) Let $N = \{ A \in A : \omega(A^*A) = 0 \}$ prove that $A \in A, N \in N$ implies that $AN \in N$.

(ii) For any $A \in A$, denote $\psi_A = \{ \tilde{A} \in A : \exists N \in N : \tilde{A} = A + N \}$ and let $h$ be the set of such equivalence classes. Prove that

$$\langle \psi_A, \psi_B \rangle = \omega(A^*B)$$

is independent of the representatives $A, B$ and defines an inner product on $h$.

(iii) Prove that the linear map $\pi : A \to \mathcal{B}(h)$ defined by

$$\pi(A)\psi_B = \psi_{AB}$$

is bounded, and a *-morphism, namely

$$\pi(A^*) = \pi(A)^*, \quad \pi(AB) = \pi(A)\pi(B).$$

(iv) Denote $\Omega = \psi_I \in h$. Prove that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle.$$

Solution. (i) Using Problem 2(i) of Sheet 1 and the C*-property, we have that

$$\omega((AN)^*AN) \leq \|A\|^2\omega(N^*N) = 0$$

proving the claim.

(ii) We first check independence of the representative. For any $N, M \in N$,

$$\omega((A + N)^*(B + M)) = \omega(A^*B) + \omega(A^*M) + \omega(B^*N) + \omega(N^*M),$$

and the three last terms vanish by the Cauchy-Schwarz inequality. Moreover, $\langle \psi_A, \psi_A \rangle = \omega(A^*A) \geq 0$ by the positivity of $\omega$, and product vanishes only on the class $\psi_0$ by construction. Finally

$$\langle \psi_A, \psi_B \rangle = \omega(A^*B) = \overline{\omega(B^*A)} = \overline{\langle \psi_B, \psi_A \rangle}.$$ 

(iii) A calculation yields boundedness:

$$\|\pi(A)\psi_B\|_h^2 = \langle \psi_{AB}, \psi_{AB} \rangle = \omega(B^*A^*AB) \leq \|A\|^2\omega(B^*B) = \|A\|^2\|\psi_B\|_h^2.$$

Moreover,

$$\langle \psi_A, \pi(B^*)\psi_C \rangle = \langle \psi_A, \psi_{B^*C} \rangle = \omega(A^*B^*C) = \langle \psi_{BA}, \psi_C \rangle = \langle \pi(B)\psi_A, \psi_C \rangle$$

showing that $\pi(B^*) = (\pi(B))^*$. Finally,

$$\pi(AB)\psi_C = \psi_{ABC} = \pi(A)\psi_{BC} = \pi(A)\pi(B)\psi_C$$

which shows that $\pi(AB) = \pi(A)\pi(B)$.

(iv) Since $\psi_A = \psi_I = \pi(A)\Omega$, we conclude that $\omega(A) = \omega(IA) = \langle \Omega, \psi_A \rangle = \langle \Omega, \pi(A)\Omega \rangle$ indeed.

Remark. Since $\mathcal{H}$ is the completion of $h$, the identity $\psi_A = \pi(A)\Omega$ shows that $\Omega$ is cyclic for $\pi(A)$ in $\mathcal{H}$, namely $\{ \pi(A)\Omega : A \in A \}$ is dense in $\mathcal{H}$. 


Problem 2. Same assumptions as in Problem 1.
(i) Prove that the GNS representation is unique up to unitary equivalence. More precisely, prove that, given any two representations \((H_i, \pi_i, \Omega_i), i = 1, 2\), construct a unitary intertwiner \(U : H_1 \rightarrow H_2\):
\[ U\pi_1(A) = \pi_2(A)U, \quad U\Omega_1 = \Omega_2. \]
(ii) Let \(\alpha\) be a \(*\)-automorphism of \(A\) and \(\omega\) be an \(\alpha\)-invariant state:
\[ \omega \circ \alpha = \omega. \]
Prove that there is a unique unitary operator \(U\) on the GNS Hilbert space \(H_\omega\) such that \(\pi_\omega(A)U = U\pi_\omega(\alpha(A))\) for \(A \in A\) and \(U\Omega_\omega = \Omega_\omega\).

Solution. (i) We define
\[ U : H_1 \rightarrow H_2 \]
\[ \pi_1(A)\Omega_1 \mapsto \pi_2(A)\Omega_2 \]
on the dense set \(\{\pi(A)\Omega : A \in A\}\). In particular, \(U\Omega_1 = \Omega_2\). Now
\[ \langle \pi_1(B)\Omega_1, (A)\Omega_1 \rangle_{H_1} = \omega(B^*A) = \langle \pi_2(B)\Omega_2, \pi_2(A)\Omega_2 \rangle_{H_2} = \langle U\pi_1(B)\Omega_1, U\pi_1(A)\Omega_1 \rangle_{H_2}, \]
so that \(U\) is an isometry. In particular, it is bounded. Since it is densely defined, it can be extended by linearity to all of \(H_1\) with the same bound. To prove surjectivity, we note that for any \(\Psi_2 \in H_2\), there is a sequence \((A_n)_{n \in \mathbb{N}}\) with \(A_n \in A\) such that \(\pi_2(A_n)\Omega_2 \rightarrow \Psi_2\). Let \(\Psi_1 = \lim_{n \rightarrow \infty} \pi_1(A_n)\Omega_1\). Then by continuity of \(U\), \(\Psi_2 = \lim_{n \rightarrow \infty} U\pi_1(A_n)\Omega_1 = U\Psi_1\). Hence \(U\) is unitary. The intertwining property follows immediately from
\[ \pi_2(A)U\pi_1(B)\Omega_1 = \pi_2(A)\pi_2(B)\Omega_2 = \pi_2(AB)\Omega_2 = U\pi_1(AB)\Omega_1 = U\pi_1(A)\pi_1(B)\Omega_1. \]
and density again.
(ii) For any \(A \in A\),
\[ \langle \Omega_\omega, \pi_\omega \circ \alpha(A)\Omega_\omega \rangle = \langle \Omega_\omega, \pi_\omega(\alpha(A))\Omega_\omega \rangle = \omega(\alpha(A)) = \omega(A) \]
which proves that \((H_\omega, \pi_\omega \circ \alpha, \Omega_\omega)\) is GNS representation for \(\omega\). The claim now follows from (i).

Remark. In the case where the state is invariant under a dynamics \(\tau_t\) (as for example when \(\omega\) is an equilibrium state), the result provides a family of unitaries \(U(t)\) such that
\[ U(t)^*\pi_\omega(A)U(t) = \pi_\omega(\tau_t(A)), \quad U(t)\Omega = \Omega. \]
Since it is strongly continuous, there exists \(L = L^*\) such that \(U(t) = \exp(-itL)\) and \(L\Omega = 0\). One says that the dynamics is unitarily implementable in the GNS Hilbert space.
Problem 3. Consider the C*-algebra $A = \mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Let be a strictly positive density matrix on $\mathcal{H}$, namely $\rho = \rho^\ast$, $0 < \rho \leq I$, $\text{Tr}(\rho) = 1$. Let

$$\omega(A) = \text{Tr}(\rho A)$$

be the corresponding state. Consider the triple $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ defined as follows:

- $\mathcal{H}_\rho$ is the set of Hilbert-Schmidt operators (namely those $T \in \mathcal{B}(\mathcal{H})$ such that $\text{Tr}(T^* T) < \infty$), with the scalar product $\langle T, S \rangle_{\mathcal{H}_\rho} := \text{Tr}(T^* S)$.
- For each $A \in \mathcal{A}$, let $\pi_\rho(A) : \mathcal{H}_\rho \to \mathcal{H}_\rho$ be defined by $\pi_\rho(A)T := AT$ for $T \in \mathcal{H}_\rho$.
- $\Omega_\rho := \rho^{1/2}$.

(i) Prove that $\Omega_\rho$ is a unit vector in $\mathcal{H}_\rho$.
(ii) Prove that $\pi_\rho$ is a faithful representation of $A$ into $\mathcal{B}(\mathcal{H}_\rho)$, namely that $\text{Ker}(\pi_\rho) = \{0\}$.
(iii) Prove that $\rho_\rho$ is a cyclic vector for the representation $\pi_\rho$, i.e. that the set $\{\pi_\rho(A)\Omega_\rho : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\rho$.
(iv) Prove that $\omega(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle$ for all $A \in \mathcal{A}$.

Solution. We denote $\mathcal{I}_2(\mathcal{H})$ the set of Hilbert-Schmidt operators. It is a two-sided ideal: If $A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{I}_2(\mathcal{H})$, then $|\text{Tr}(B^* A^\ast AB)| \leq ||A||^2 \text{Tr}(B^* B) < \infty$ and similarly for $\text{Tr}(A^\ast B^* BA) = \text{Tr}(B A A^* B^*)$. This in particular shows that $\pi_\rho$ is well-defined.

(i) Immediate from $||\Omega_\rho||^2 = \text{Tr}(\rho^{1/2} \rho^{1/2}) = 1$.
(ii) Let $A \in \mathcal{B}(\mathcal{H})$ be such that $\pi_\rho(A) = 0$. Then for any $T \in \mathcal{I}_2(\mathcal{H})$, $AT = \pi_\rho(A)T = 0$ and hence

$$0 = ||\pi_\rho(A)||_{\mathcal{I}_2(\mathcal{H})} = \sup_{||T||_{\mathcal{I}_2(\mathcal{H})} = 1} |\langle T, \pi_\rho(A) \rangle_{\mathcal{I}_2(\mathcal{H})}| = \sup_{||T||_{\mathcal{I}_2(\mathcal{H})} = 1} |\text{Tr}(\pi_\rho(A)T^* )| = \sup_{||T||_{\mathcal{I}_2(\mathcal{H})} = 1} |\text{Tr}(AT^* )|$$

which implies that $A = 0$.

(iii) Let $B \in \mathcal{I}_2(\mathcal{H})$. Since $\text{Ker} \rho = \{0\}$, it is invertible so that $B \rho^{-1/2} \in \mathcal{I}_2(\mathcal{H})$.

$$\pi_\rho(B \rho^{-1/2})\Omega_\rho = \pi_\rho(B \rho^{-1/2})\rho^{1/2} = B \rho^{-1/2} \rho^{1/2} = B,$$

proving cyclicity.

(iv) Immediate from the definitions:

$$\langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle = \text{Tr}(\rho^{1/2} A \rho^{1/2}) = \text{Tr}(\rho A) = \omega(A).$$

Remark. Given a dynamics $\tau_t(A) = e^{itH} A e^{-itH}$ for which $[\rho, H] = 0$, the generator $L$ discussed at the end of Problem 2 is given by

$$e^{-itL} B = e^{itH} B e^{-itH} \quad (B \in \mathcal{I}_2(\mathcal{H}))$$

namely $L = [H, \cdot]$. Indeed

$$e^{itL} \pi_\rho(A) e^{-itL} B = e^{itH} (A e^{-itH} B e^{itH}) e^{-itH} = (e^{itH} A e^{-itH}) B = \pi_\rho(\tau_t(A)) B,$$

and $L \Omega_\rho = [H, \rho^{-1/2}] = 0$. 
**Problem 4.** Let $\mathcal{A}$ be the CAR algebra over the Hilbert space $\mathcal{H}$.

(i) Prove that $\|b(f)\| = \|b^*(f)\| = \|f\|_H$.

(ii) Compute the spectrum of $b(f), b^*(f)$ and $b^*(f)b(f)$.

(iii) Let now $\mathcal{H} = l^2(\Lambda)$, where $\Lambda$ is a finite set and let $N = \sum_{x \in \Lambda} b^*_x b_x$ where $b_x = b(\delta_x)$ acting on Fock space. Prove that $\sigma(N) = \{0, 1, \ldots |\Lambda|\}$.

(iv) Prove that for any $\theta \in \mathbb{R}$,

$$e^{i\theta N} b(f) e^{-i\theta N} = b(e^{i\theta f}).$$

(v) Let $A^+_\Lambda$ be the even subalgebra of $\mathcal{A}$ and let $A \in A^+_\Lambda, B \in A_Y$. Prove that $[A, B] = 0$ whenever $X \cap Y = \emptyset$.

**Solution.** (i) By the CAR relations,

$$(b^*(f)b(f))^2 = b^*(f)\{b(f), b^*(f)\}b(f) = \|f\|_H^2 b^*(f)b(f)$$

and the C*-property of the norm implies that $\|b(f)\|^4 = \|f\|_H^2 \|b(f)\|^2$. Hence $\|b(f)\| = \|f\|_H$ which concludes the proof since $\|b(f)\| = \|b^*(f)\|$. 

(ii) The CAR relations imply that $(b(f))^2 = (b^*(f))^2 = 0$. In other words, they are nilpotent and hence $\sigma(b(f)) = \sigma(b^*(f)) = \{0\}$. Furthermore, $n(f) = b^*(f)b(f)$ is a non-negative self-adjoint element and (1) reads $n(f)(\mathbb{I} - n(f)) = 0$. It follows that $\sigma(n(f)) \subset \{0, 1\}$. We also have that $n(f)b^*(f) = b^*(f)$ so that $n(f) \neq 0$ and similarly $(\mathbb{I} - n(f))b(f) = b(f)$ so that $\mathbb{I} - n(f) \neq 0$. It follows that $\sigma(n(f)) = \{0, 1\}$.

(iii) We check that $n_x b^*_y \Omega = \delta_{x,y} b^*_x \Omega$. Moreover, the vectors

$$\prod_{i=1}^k b^*_{x_i} \Omega, \quad x_i \neq x_j \text{ if } i \neq j, 0 \leq k \leq |\Lambda|,$$

are such that

$$N_{\Lambda} \prod_{i=1}^k b^*_{x_i} \Omega = \sum_{j=1}^k n_{x_j} \prod_{i=1}^k b^*_{x_i} \Omega = \prod_{i=1}^k b^*_{x_i} \Omega,$$

so that $\sigma(N_{\Lambda}) \supset \{0, 1, \ldots, |\Lambda|\}$. Since these $\sum_{k=0}^{|\Lambda|} \binom{|\Lambda|}{k} = 2^{|\Lambda|} = \dim(F_\Lambda)$ vectors are furthermore mutually orthogonal they form a basis and $\sigma(N_{\Lambda}) = \{0, 1, \ldots, |\Lambda|\}$.

(iv) We first note that

$$[N_{\Lambda}, b(f)] = \sum_{x \in \Lambda} [n_x, b(f)] = \sum_{x \in \Lambda} -f(x)b_x = -b(f).$$

Since $N_{\Lambda}$ is bounded, $e^{-i\theta N_{\Lambda}}$ is uniformly differentiable and

$$\frac{d}{d\theta} e^{-i\theta N_{\Lambda}} b(f) e^{i\theta N_{\Lambda}} = e^{-i\theta N_{\Lambda}} [-iN_{\Lambda}, b(f)] e^{i\theta N_{\Lambda}} = ie^{-i\theta N_{\Lambda}} b(f) e^{i\theta N_{\Lambda}}.$$

Similarly,

$$\frac{d}{d\theta} b(e^{-i\theta f}) = \frac{d}{d\theta} e^{i\theta b(f)} = ib(e^{-i\theta f})$$

Since the two functions $\theta \mapsto e^{-i\theta N_{\Lambda}} b(f) e^{i\theta N_{\Lambda}}$ and $\theta \mapsto b(e^{-i\theta f})$ satisfy the same ODE with the initial condition $b(f)$ they are equal.

(v) By the above,

$$e^{-i\theta N_{\Lambda}} b^*_x e^{i\theta N_{\Lambda}} = -b^*_x, \quad e^{-i\theta N_{\Lambda}} n_x e^{i\theta N_{\Lambda}} = n_x.$$
where $b^\sharp_x$ refers to either $b_x$ or $b^*_x$. The monomials

$$\prod_{i=1}^{k} B_{x_i}, \quad B_x \in \{b_x, b^*_x, n_x\}, x_i \neq x_j \text{ if } i \neq j, 0 \leq k \leq |\Lambda|,$$

for a vector space basis of $\mathcal{A}_\Lambda$ and they are even if and only if the product contains an even number of $b^\sharp$. If $A$ is such a monomial and $B$ is an arbitrary monomial, the fact that they have disjoint support implies that any $b^\sharp$ in $B$ anticommutes with any one of $A$. Since there is an even number of them in $A$, any $b^\sharp$ in $B$ commutes with $A$. We conclude recursively using Leibnitz’ rule for the commutator.