Problem 1. Let $\Gamma$ be a countable set equipped with a metric $d$. We say that $(\Gamma, d)$ is $\nu$-regular if there is $\kappa < \infty$ such that
\[
\sup_{x \in \Gamma} |B_x(n)| \leq \kappa n^\nu
\]
for all $n \in \mathbb{N}$, where $B_x(n) = \{ y \in \Gamma : d(x, y) \leq n \}$.
(i) Prove that $\mathbb{Z}^\nu$ equipped with the graph distance is $\nu$-regular for any $\nu \in \mathbb{N}$.
(ii) Prove that if there is an $F$-function such that $F(r) \geq \frac{C}{r^\nu}$, then $(\Gamma, d)$ is $\nu$-regular.
(iii) Let $(\Gamma, d)$ be $\nu$-regular. Prove that $F_g$ defined by
\[
F_g(r) = e^{-g(r)} F(r)
\]
is an $F$-function with convolution constant $C_{F_g} \leq C_F$, $\|F_g\| \leq \|F\|$ whenever $F$ is.

Solution. (i) In $\mathbb{Z}^\nu$, the ball is a ‘diamond’ which is contained in a hypercube of side length $2n+1$, yielding $|B_x(n)| \leq (2n + 1)^\nu$ indeed.
(ii) We first note that
\[
F(n)|B_x(n)| \leq \sum_{y \in B_x(n)} F(d(x, y)) \leq \sum_{y \in \Gamma} F(d(x, y)) \leq \|F\|.
\]
Hence $|B_x(n)| \leq Cn^\nu$ by assumption.
(iii) The decomposition $\sum_{y \in \Gamma} f(y) = f(x) + \sum_{n=1}^{\infty} \sum_{y \in B_x(n) \setminus B_x(n-1)}$ yields
\[
\sum_{y \in \Gamma} \frac{1}{(1+d(x, y))^{\nu+1+\epsilon}} \leq 1 + \sum_{n=1}^{\infty} \frac{|B_x(n)|}{n^{\nu+1+\epsilon}} \leq 1 + \sum_{n=1}^{\infty} \frac{\kappa}{n^{1+\epsilon}}
\]
uniformly in $x \in \Gamma$. Since this is convergent, $\|F\| < \infty$ indeed. As for the convolution condition, we first note that $1 + d(x, z) \leq 1 + d(x, y) + 1 + d(y, z)$ and since $\xi \mapsto \xi^p$ is increasing and convex for $p \geq 1$, we conclude that
\[
(1 + d(x, z))^p \leq 2^p \left( \frac{1}{2} (1 + d(x, y)) + \frac{1}{2} (1 + d(y, z)) \right)^p
\]
\[
\leq 2^{p-1} \left( (1 + d(x, y))^p + (1 + d(y, z))^p \right).
\]
Applying this to $p = \nu + \epsilon + 1$, we obtain,

$$\sum_{y \in \Gamma} \frac{F(d(x, y))F(d(y, z))}{F(d(x, z))} \leq 2^{\nu+\epsilon} \sum_{y \in \Gamma} F(d(x, y))F(d(y, z)) \left( \frac{1}{F(d(x, y))} + \frac{1}{F(d(x, y))} \right) \leq 2^{\nu+\epsilon}\|F\|$$

indeed.

(iv) Summability of $F_g$ is an immediate consequence of $g \geq 0$ by $e^{-g(d(x,y))} \leq 1$, which in turn yields $\|F_g\| \leq \|F\|$. The convolution condition follows from monotonicity and subadditivity of $g$

$$g(d(x, y)) \leq g(d(x, y) + d(y, z)) \leq g(d(x, y)) + g(d(y, z)),$$

which yields

$$\frac{e^{-g(d(x,y))} - g(d(y,z))}{e^{-g(d(y,z))}} \leq 1$$

and in turn $C_{F_g} \leq C_F$. 
**Problem 2.** Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where both factors are finite dimensional. Recall that $\text{Tr}_1(A)$ is the unique element of $B(\mathcal{H}_2)$ such that

$$\text{Tr}(A(\mathbb{I} \otimes B)) = \text{Tr}(\text{Tr}_1(A)B)$$

for all $B \in B(\mathcal{H}_2)$.

(i) Prove that $\text{Tr}_1$ is trace and positivity preserving:

(a) $\text{Tr}(\text{Tr}_1(A)) = \text{Tr}(A)$,

(b) $A \geq 0 \implies \text{Tr}_1(A) \geq 0$.

Conclude that if $\rho$ is a density matrix on $\mathcal{H}$, then $\text{Tr}_1(\rho)$ is a density matrix on $\mathcal{H}_2$.

(ii) Let now $\rho$ be a pure state on $\mathcal{H}$. Prove that $\text{Tr}_1(\rho)$ and $\text{Tr}_2(\rho)$ have the same non-zero spectrum, including multiplicities.

**Solution.** (i) (a) Follows from $\text{Tr}(\text{Tr}_1(A)(\mathbb{I} \otimes \mathbb{I})) = \text{Tr}(A\mathbb{I}) = \text{Tr}(A)$ by definition. Let $B \in B(\mathcal{H}_2)$ be such that $B \geq 0$. Then $\mathbb{I} \otimes B \geq 0$ and so

$$0 \leq \text{Tr}(A(\mathbb{I} \otimes B)) = \text{Tr}(\text{Tr}_1(A)B)$$

since $A$ is positive. Choosing $B$ to be the one-dimensional projector on any $v \in \mathcal{H}_2$, we conclude that $\langle v, \text{Tr}_1(A)v \rangle \geq 0$ for all $v \in \mathcal{H}_2$, so that $\text{Tr}_1(A) \geq 0$.

If $\rho$ is a density matrix on $\mathcal{H}$, it follows that $\text{Tr}_1(\rho)$ has unit trace and that $\text{Tr}_1(\rho) \geq 0$. Hence $\text{Tr}_1(\rho)$ is a density matrix on $\mathcal{H}_2$.

(ii) If $\rho$ is a pure state, it is a one-dimensional projection $\rho = P_\psi$. Then for any $A \in B(\mathcal{H}_2)$,

$$\text{Tr}(\text{Tr}_1(\rho)A) = \text{Tr}(\rho(\mathbb{I} \otimes A)) = \langle \psi, (\mathbb{I} \otimes A)\psi \rangle.$$

Picking bases in $\mathcal{H}_1, \mathcal{H}_2$, we write this as

$$\langle \psi, (\mathbb{I} \otimes A)\psi \rangle = \sum_{i,j,k,l} \overline{\psi}_{ij}(\mathbb{I} \otimes A)_{ij,kl}\psi_{kl} = \sum_{i,k,l} \overline{\psi}_{ij}A_{jl}\psi_{il} = \text{Tr}((\Psi^*\Psi)^tA)$$

where $\Psi$ is the matrix of the coefficients of the vector $\psi$. A similar calculation yields that for any $B \in B(\mathcal{H}_1)$,

$$\text{Tr}(\text{Tr}_2(\rho)B) = \text{Tr}((\Psi^*\Psi^t)B)$$

By unicity of the partial trace, we see that

$$\text{Tr}_2(\rho) = \Psi\Psi^*, \quad \text{Tr}_1(\rho) = (\Psi^*\Psi)^t,$$

which have the same non-zero spectrum. Indeed, if $MM^*v = \lambda v$ with $\lambda \neq 0$, then in particular $w = M^*v \neq 0$ is an eigenvector of $M^*M$ for the same eigenvalue applying $M^*$ to the eigenvalue equation above. Moreover, if $v_1, v_2$ are linearly independent eigenvectors of $MM^*$ for the same eigenvalue $\lambda$, then $0 = \mu_1w_1 + \mu_2w_2 = M^*(\mu_1v_1 + \mu_2v_2)$ implies by applying $M$ that $0 = \mu_1v_1 + \mu_2v_2$ since $\lambda \neq 0$ and hence $w_1, w_2$ are linearly independent.
**Problem 3.** Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where both factors are finite dimensional.

(i) Let $\mathcal{U}_1 = \{U \in \mathcal{B}(\mathcal{H}_1) : U^* U = UU^* = I\}$ be the unitary group over $\mathcal{H}_1$. Prove that

$$I \otimes \text{Tr}_1(A) = \dim(\mathcal{H}_1) \int_{\mathcal{U}_1} (U \otimes I)^* A(U \otimes I) d\mu(U)$$

where the measure $\mu$ is the normalized Haar measure on $\mathcal{U}_1$.

(ii) Prove that if $A \in \mathcal{B}(\mathcal{H})$ is such that $\|[A, B \otimes I]\| \leq \epsilon \|B\|$ for all $B \in \mathcal{B}(\mathcal{H}_1)$, then there exists $\tilde{A} \in \mathcal{B}(\mathcal{H}_2)$ such that

$$\|A - I \otimes \tilde{A}\| \leq \epsilon.$$

**Solution.** (i) By the invariance of the Haar measure, the proposed integral is invariant under conjugation by any $V \otimes I$, where $V \in \mathcal{U}_1$. By Schur’s lemma,

$$\int_{\mathcal{U}_1} (U \otimes I)^* A(U \otimes I) d\mu(U) = I \otimes \tilde{A}$$

and it remains to determine $\tilde{A}$. Since $U \otimes I$ commutes with $I \otimes I$ (because $(U \otimes I)(I \otimes B) = U \otimes B = (I \otimes B)(U \otimes I)$), unitary invariance of the trace implies that

$$\text{Tr}(A(I \otimes B)) = \int_{\mathcal{U}_1} \text{Tr}((U \otimes I)^* A(U \otimes I)(I \otimes B)) d\mu(U)$$

and by the remark above,

$$\text{Tr}(A(I \otimes B)) = \text{Tr}((I \otimes \tilde{A})(I \otimes B)) = \dim(\mathcal{H}_1) \text{Tr}(\tilde{A}B)$$

namely $\dim(\mathcal{H}_1) \tilde{A} = \text{Tr}_1(A)$ by uniqueness of the partial trace.

(ii) With $\tilde{A}$ introduced above,

$$A - I \otimes \tilde{A} = \int_{\mathcal{U}_1} (U \otimes I)^* [U \otimes I, A] d\mu(U).$$

This, the estimate

$$\|(U \otimes I)^*[U \otimes I, A]\| \leq \|[U \otimes I, A]\| \leq \epsilon,$$

which follows by assumption and $\|U\| = 1$, and the normalization of the Haar measure yield the claim.
Problem 4. Consider a quantum spin system on a countable set $\Gamma$, with a local interaction $\Phi \in B_F$ for some $F$-function. Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence such that
(a) $\Lambda_n \subset \Gamma$ with $|\Lambda_n| < \infty$ for all $n$
(b) $\Lambda_n \subset \Lambda_m$ whenever $n < m$
(c) for any $x \in \Gamma$, there is $n_x$ such that $x \in \Lambda_{n_x}$.

Let
$$H_n = \sum_{X \subset \Lambda_n} \Phi(X).$$

(i) For any $A \in \mathcal{A}, t \in \mathbb{R}$, let $\tau^n_t(A) = e^{i t H_n} A e^{-i t H_n}$. Prove that $(\tau^n_t(A))_{n \in \mathbb{N}}$ is a Cauchy sequence.
(ii) For any $A \in \mathcal{A}$ and $t \in \mathbb{R}$, let $\tau_t(A) = \lim_{n \to \infty} \tau^n_t(A)$. Prove that $t \mapsto \tau_t(A)$ is continuous.

Solution. (i) We assume without loss that $t > 0$. Let $\Lambda$ be a finite set and $A \in \mathcal{A}_\Lambda$. Let $m < n$ be such that $\Lambda \subset \Lambda_m$. Then
$$\tau^n_t(A) - \tau^m_t(A) = \int_0^t \frac{d}{ds}(\tau^n_s \circ \tau^m_{t-s}(A)) ds = \int_0^t \tau^n_s \left(i[H_n - H_m, \tau^m_{t-s}(A)]\right) ds.$$ 

But
$$H_n - H_m = \sum_{X \subset \Lambda_m : X \cap (\Lambda_n \setminus \Lambda_m) \neq \emptyset} \Phi(X),$$
so that
$$\|\tau^n_t(A) - \tau^m_t(A)\| \leq \sum_{x \in \Lambda_n \setminus \Lambda_m} \sum_{X \ni x} \int_0^t \|\Phi(X), \tau^m_{t-s}(A)\| ds,$$

because $\tau^n_s$ is an automorphism. We apply the Lieb-Robinson bound to the right hand side to get
$$\|\tau^n_t(A) - \tau^m_t(A)\| \leq \frac{2\|A\|}{C_F} \left(\int_0^t g_F(s) ds\right) \sum_{x \in \Lambda_n \setminus \Lambda_m} \sum_{X \ni x} \|\Phi(X)\| \|D(X, \Lambda)\|,$$

where $g_F(s) = \exp(2\|\Phi\| F C_F s)$. It remains to estimate the last factor $\sum_{X \ni x} \|\Phi(X)\| \sum_{y \in X} \sum_{z \in \Lambda} F(d(y, z))$.

We bound $\sum_{X \ni x}$ above by $\sum_{y \in \Gamma} \sum_{X \ni x, y} \|\Phi(X)\| \|F(d(x, y))\| F(d(y, z)) \leq C_F \|\Phi\| \|F\| \sum_{z \in \Lambda} F(d(x, z))$,

where we used the convolution property of $F$ and $\Phi \in B_F$. Hence,
$$\|\tau^n_t(A) - \tau^m_t(A)\| \leq 2\|A\| \|\Phi\| \|F\| \sup_{x \in \Gamma} \sum_{x \in X \setminus \Lambda_m} F(d(x, z))$$

which converges to zero as $m \to \infty$ since $F$ is integrable. This proves that $(\tau^n_t)_{n \in \mathbb{N}}$ is a Cauchy sequence for any $A \in \mathcal{A}_\Gamma^\text{loc}$ and $t \in \mathbb{R}$ fixed. If $A \in \mathcal{A}_\Gamma$ is not in $\mathcal{A}_\Gamma^\text{loc}$, there is a sequence $(A_k)_{k \in \mathbb{N}}$ with $A_k \in \mathcal{A}_\Gamma^\text{loc}$ and $A_k \to A$ as $k \to \infty$ (and in particular, $|A_k|$ is uniformly bounded). Then,
$$\|\tau^n_t(A) - \tau^m_t(A)\| \leq \|\tau^n_t(A - A_k)\| + \|\tau^n_t(A_k) - \tau^m_t(A_k)\|.$$
The first term is equal to \( \| A - A_k \| \) and it converges to 0 as \( k \to \infty \), uniformly in \( n \), while the second term converges to 0 as \( m, n \to \infty \) by the above, uniformly in \( k \).

(ii) By construction, \( t \mapsto \tau^n_t(A) \) is continuous for any \( n \in \mathbb{N} \). Moreover, the sequence \( (\tau^n_t(A))_{n \in \mathbb{N}} \) is Cauchy, uniformly in \( t \) for \( t \) in a compact set. Hence \( t \mapsto \tau_t(A) \) is continuous on \( \mathbb{R} \).

**Remarks.** (a) It is elementary to check that \( \{ \tau_t : t \in \mathbb{R} \} \) is a strongly continuous group of \(*\)-automorphisms of \( A \). We have therefore proved the existence of the dynamics in the infinite volume limit.

(b) A similar argument would show that the map \( \Phi \mapsto \tau_t^\Phi(A) \) is continuous with respect to the topology of the Banach space \( B_F \). In other words, the dynamics of observables depends continuously on the interaction.