Problem 1. Let $A$ be a C*-algebra and $A \in A$.

(i) Prove that if $\lambda \in \mathbb{C}$ is such that $|\lambda| > \|A\|$, then the sum

$$\sum_{n=0}^{\infty} \left( \frac{A}{\lambda} \right)^n$$

is convergent and that $\lambda I - A$ is invertible (with inverse in $A$).

(ii) The spectrum $\sigma(A)$ of $A \in A$ is the set $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not invertible. Prove that if $A = A^*$ then $\sigma(A) \subseteq [-\|A\|, \|A\|]$.

(iii) Let $\mathcal{I} \subset A$ be the set of invertible elements of $A$. Prove that $\mathcal{I}$ is an open set and that the map $A \mapsto A^{-1}$ is continuous on $\mathcal{I}$.

Hint. $B = A(I - A^{-1}(A - B))$ whenever $A \in \mathcal{I}$.

Solution. (i) We compute

$$\left\| \sum_{n=0}^{N} \left( \frac{A}{\lambda} \right)^n \right\| \leq \sum_{n=0}^{N} \left( \frac{\|A\|}{|\lambda|} \right)^n$$

which is convergent iff $\|A\| < |\lambda|$. In that case,

$$(\lambda I - A) \frac{1}{\lambda} \sum_{n=0}^{N} \left( \frac{A}{\lambda} \right)^n = I - \left( \frac{A}{\lambda} \right)^{N+1} \rightarrow I \quad (N \rightarrow \infty),$$

in norm, proving that $\frac{1}{\lambda} \sum_{n=0}^{\infty} \left( \frac{A}{\lambda} \right)^n$ is the right inverse of $\lambda I - A$. The same calculation shows that it is also its left inverse.

Remark. In this context, the series is called the Neumann series.

(ii) Let now $A = A^*$ and let $\lambda = x + iy \in \sigma(A)$, with $x, y \in \mathbb{R}$ and $x^2 + y^2 \leq \|A\|^2$ by (i). We claim that $y = 0$. Let $t \in \mathbb{R}$. By the C*-property and $A = A^*$,

$$\|(A + itI)\|^2 = \|(A + itI)(A - itI)\| = \|A^2 + t^2I\| \leq \|A\|^2 + t^2.$$ 

Since $\lambda + it \in \sigma(A + itI)$, the inequality above and (i) imply that

$$|x + iy + t|^2 \leq \|A\|^2 + t^2,$$

namely

$$2yt \leq \|A\|^2 - x^2 - y^2.$$ 

Since this holds for all $t \in \mathbb{R}$, we must have that $y = 0$.

(iii) Let $A \in \mathcal{I}$ and let $B$ be such that $\|B - A\| < \frac{1}{\|A^{-1}\|}$. By (i), it follows that $I - A^{-1}(A - B)$ is
invertible, so that $B$ is invertible and hence $\mathcal{I}$ is open. Furthermore, again by (i)

$$
\|A^{-1} - B^{-1}\| = \left\| A^{-1} - \sum_{n=0}^{\infty} (A^{-1}(A - B))^n A^{-1} \right\|
$$

$$
\leq \|A^{-1}\| \sum_{n=1}^{\infty} \|A^{-1}(A - B)\|^n
$$

$$
= \frac{\|A^{-1}\|^2 \|A - B\|}{1 - \|A^{-1}\| \|A - B\|} < 2\|A^{-1}\|^2 \|A - B\|
$$

if $\|A - B\| < \frac{1}{2\|A^{-1}\|}$, proving continuity.
Problem 2. Let $\mathcal{A}$ be a $C^*$-algebra and $\omega : \mathcal{A} \to \mathbb{C}$ be a state.

(i) Prove the Cauchy-Schwarz inequality: For any $A, B \in \mathcal{A}$,

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B),$$

as well as

$$\omega(A^*B) = \omega(B^*A)$$

and

$$|\omega(A^*BA)| \leq \|B\|\omega(A^*A).$$

Hint. You may need the following facts about positive elements of a $C^*$-algebra. A $\in \mathcal{A}$ is called positive if $A = A^*$ and $\sigma(A) \subset [0, \infty)$. In that case, the integral

$$\pi \int_0^\infty \sqrt{\lambda} (\lambda^{-1} - (\lambda I + A)^{-1}) \, d\lambda$$

is convergent and defines a self-adjoint operator $S$ such that $S^2 = S^*S = A$. Reciprocally, any element of the form $A^*A$ is a positive element of the algebra.

(ii) Let $\text{Var}_\omega(A) = \omega((A - \omega(A)I)^*(A - \omega(A)I))$. Prove the Heisenberg inequality

$$\text{Var}_\omega(A)\text{Var}_\omega(B) \geq \frac{1}{4} |\omega([A, B])|^2, \quad [A, B] = AB - BA$$

for any self-adjoint elements $A = A^* \in \mathcal{A}, B = B^* \in \mathcal{A}$.

Solution. (i) Since the quadratic form

$$\mathbb{C} \ni \lambda \mapsto \omega((A + \lambda B)^*(A + \lambda B)) = \omega(A^*A) + \overline{\lambda} \omega(B^*A) + \lambda \omega(A^*B) + |\lambda|^2 \omega(B^*B)$$

is non-negative, it is in particular real so that $\omega(B^*A) - \overline{\omega(A^*B)} = 0$, which is the second equality. Choosing $\lambda \in \mathbb{R}$, non-negativity implies that the discriminant is always non-positive,

$$4\text{Re}(\omega(A^*B))^2 - 4\omega(B^*B)\omega(A^*A) \leq 0,$$

which yields the Cauchy-Schwarz inequality by noting that $|\text{Re}(z)| \leq |z|$. To prove the last one, we first apply Cauchy-Schwarz to obtain

$$|\omega(A^*BA)|^2 \leq \omega(A^*A)\omega(A^*B^*BA). \quad (1)$$

We then note that the inequality $A^*B^*BA \leq \|B\|^2 A^*A$ together with the positivity and linearity of the state implies that

$$\omega(A^*B^*BA) - \|B\|^2 \omega(A^*A) = \omega(A^*B^*BA - \|B\|^2 A^*A) \leq 0$$

which yields the claim when used in (1). Note that $\sigma(B^*B - \|B\|^2 I) = \sigma(B^*B) - \|B\|^2 I \subset [-2\|B\|^2, 0]$ implies that $B^*B - \|B\|^2 I \leq 0$. In particular, there is $D$ such that $B^*B - \|B\|^2 I = -D^*D$. But then $A^*B^*BA - \|B\|^2 A^*A = -(A^*D^*DA = -(DA)^*(DA) \leq 0$.

(ii) It suffices to apply the inequality

$$\omega(X^*X)\omega(Y^*Y) \geq |\omega(X^*Y)|^2 \geq |\text{Im}(\omega(X^*Y))|^2 = \frac{1}{4} |\omega(X^*Y - Y^*X)|^2 \quad (2)$$

to the self-adjoint $X = A - \omega(A), Y = B - \omega(B)$ since $[X, Y] = [A, B]$.

Remark. If $\mathcal{A}$ is an Abelian algebra, then the right hand side of the inequality vanishes for all $A, B \in \mathcal{A}$. This is the case for classical systems.
Problem 3. Let $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C})$ be the space of square-integrable functions with scalar product and norm
\[
\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx, \quad \|f\| = \left( \int_{\mathbb{R}} |f(x)|^2dx \right)^{\frac{1}{2}}.
\]

Let $\mathcal{S}$ be the space of Schwartz functions:
\[
\mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^n \frac{d^m}{dx^m} f(x)| < \infty \text{ for all } n, m \in \mathbb{N} \right\} \subset \mathcal{H}.
\]

Finally let $Q, P$ be defined by
\[
(Qf)(x) = xf(x), \quad (Pf)(x) = -i f'(x).
\]

(i) Prove that $QS \subset \mathcal{S}$ and $PS \subset \mathcal{S}$.
(ii) For any $f \in \mathcal{S}$ with $\|f\| = 1$, let $\omega_f(\cdot) = \langle f, (\cdot)f \rangle$. Prove that
\[
\text{Var}_{\omega_f}(Q)\text{Var}_{\omega_f}(P) \geq \frac{1}{4}.
\]
(iii) Prove that equality holds above if and only if
\[
f(x) = ce^{-ipx}e^{-\xi(x+q)^2},
\]
where $c \in \mathbb{C}, \xi > 0$ and $p, q \in \mathbb{R}$.

Hint. Consider first the case $\omega_f(Q) = \omega_f(P) = 0$.

Solution. (i) Immediate by the definition of a Schwartz function.
(ii) We first note that $Q, P$ are symmetric operators on $\mathcal{S}$, namely
\[
\langle f, Qf \rangle = \langle Qf, f \rangle, \quad \langle f, Pf \rangle = \langle Pf, f \rangle,
\]
for all $f \in \mathcal{S}$, where the second equality holds by integration by parts and the fast decay of Schwartz functions at infinity. (i) and the Cauchy-Schwarz inequality for the $L^2$-scalar product \(|\langle f, g \rangle| \leq \|f\|\|g\|\) imply
\[
|\omega_f(Q^*P)|^2 = |\langle Qf, Pf \rangle|^2 \leq \|Qf\|^2\|Pf\|^2 = \omega_f(Q^*Q)\omega_f(P^*P),
\]
namely the eponymous inequality for the state $\omega_f$. In particular, the Heisenberg inequality holds for $f \in \mathcal{S}$ with
\[
\omega_f([Q, P]) = \langle f, Q Pf \rangle - \langle f, PQf \rangle = -i \int_{\mathbb{R}} f(x)\left(xf'(x) - \frac{d}{dx}(xf(x))\right)dx = i\|f\|^2 = i.
\]
(note that all operations are well-defined by (i) and $f \in \mathcal{S}$). We conclude as in Problem 2 that
\[
\text{Var}_{\omega_f}(Q)\text{Var}_{\omega_f}(P) \geq \frac{1}{4},
\]
which is the celebrated Heisenberg’s Uncertainty Principle.

Remark. This setting is that of a single particle moving on the line. The measure $d\rho(x) = |f(x)|^2dx$ has the physical interpretation of the probability density that the particle is around $x$. Its first
moment, which is \( \langle f, Qf \rangle \), is the expectation value of the position of the particle. After Fourier transformation, \( d \mu(k) = |\hat{f}(k)|^2 dk \) is the probability density for the momentum of the particle. Hence, Heisenberg's inequality is a lower bound on the product of the variances of position and momentum of the particle, which is independent of the wavefunction \( f \).

(iii) It suffices to consider the case \( \omega_f(Q) = \omega_f(P) = 0 \). Indeed, if \( p = \omega_f(P), q = \omega_f(Q) \), these expectation values vanish for the function \( x \mapsto e^{-ipx} f(x+q) \). Equality holds in Heisenberg's bound iff the following equalities hold:

\[
|\text{Im} \langle Qf, Pf \rangle| = |\langle Qf, Pf \rangle|, \quad |\langle Qf, Pf \rangle| = \|Qf\| \|Pf\|,
\]

see (2,3), namely

\[
\langle Qf, Pf \rangle \in i\mathbb{R}, \quad Qf = \mu Pf \text{ for } \mu \in \mathbb{C}.
\]

By the second condition, \( f \) solves the differential equation

\[
f'(x) = \frac{i}{\mu} xf(x)
\]

namely \( f(x) = ce^{\frac{ix^2}{2\mu}} \). Now

\[
\langle Qf, Pf \rangle = \frac{|c|^2}{\mu} \int_{\mathbb{R}} xe^{ix^2} dx \in i\mathbb{R}
\]

iff \(-\xi = \frac{i}{\mu} < 0\), namely \( f_0(x) = ce^{-\xi x^2} \). By the initial remark, the general solution is given by

\[
f_{p,q}(x) = e^{-ipx} f_0(x + q) \quad (p, q \in \mathbb{R}).
\]
Problem 4. Let $\mathcal{H}$ be a finite-dimensional Hilbert space.

(i) Prove that there is a one-to-one correspondence between states (positive, linear functionals over $\mathcal{B}(\mathcal{H})$ such that $\omega(\mathbb{I}) = 1$) and density matrices (positive elements of $\mathcal{B}(\mathcal{H})$ with unit trace).

(ii) A state $\omega$ is said to be pure if it cannot be decomposed as a convex combination of other states, namely

$$\omega = \lambda \omega_1 + (1 - \lambda) \omega_2, \; \lambda \in (0, 1) \implies \omega = \omega_1 = \omega_2.$$ 

Prove that $\omega$ is pure if and only if the corresponding density matrix is a one-dimensional projection. Hint. Spectral theorem.

(iii) Let now $\mathcal{H} = \mathbb{C}^2$ and let $\mathcal{E}_2$ be the set of states over $\mathcal{H}$. Prove that there is a bijection $\mathcal{E}_2 \mapsto \{ x \in \mathbb{R}^3 : \| x \| \leq 1 \}$ such that $\omega \in \mathcal{E}_2$ is pure if and only if $\| b(\omega) \| = 1$.

Hint. Pauli matrices.

Solution. (i) Since $\mathcal{H}$ is finite dimensional, so is $\mathcal{B}(\mathcal{H})$ and so is the set of linear functionals over $\mathcal{B}(\mathcal{H})$. Hence a state is completely characterized by its action on the canonical basis $\{ e^{ij} : 1 \leq i, j \leq \dim(\mathcal{H}) \}$ and the claim follows from the equality

$$\omega(e^{ij}) = \rho_{j,i}$$

together with the fact that $\omega(\mathbb{I}) = \sum_{i=1}^{\dim(\mathcal{H})} \omega(e^{ii}) = \text{Tr}(\rho)$, which shows the equivalence of the trace condition for $\rho$ and the normalization of the state $\omega$.

(ii) We consider the corresponding density matrix $\rho$ and its spectral decomposition

$$\rho = \sum_{i=1}^{\dim(\mathcal{H})} \rho_i P_{\psi_i}$$

where $P_{\phi}$ is the one-dimensional projection onto the span of $\phi$, $\{ \rho_i : 1 \leq i \leq \dim(\mathcal{H}) \}$ are the eigenvalues and $\{ \psi_i : 1 \leq i \leq \dim(\mathcal{H}) \}$ is a corresponding orthonormal set. Since $P_{\psi_i}$ is itself a density matrix, and $1 = \text{Tr}(\rho) = \sum_i \rho_i$, the spectral decomposition provides a convex decomposition of the state $\omega$. Hence, if $\omega$ is pure, then $\rho_i = 1$, $\rho_i = 0 \; (i \neq i_0)$ and $\rho = P_{\psi_{i_0}}$ indeed. Reciprocally, if $\rho = P_{\psi_i}$, all but one eigenvalues of $\rho$ vanish and the claim follows by uniqueness of the spectral decomposition.

(iii) We express a density matrix in $\mathcal{M}_2(\mathbb{C})$ in the basis given by the identity and the three Pauli matrices:

$$\rho = \frac{1}{2} \left( a \mathbb{I} + \sum_{j=1}^{3} b_j \sigma^j \right). \quad (4)$$

Since the latter are traceless, we have that $a = 1$ by the normalization condition. $\rho$ is self-adjoint iff $b_j \in \mathbb{R}$. The eigenvalues of $b \cdot \sigma = \sum_{j=1}^{3} b_j \sigma^j$ are given by $\pm \| b \|$ so that those of $\rho$ are given by $\frac{1}{2} (1 \pm \| b \|)$. Hence $0 \leq \rho \leq \mathbb{I}$ if and only if $\| b \| \leq 1$. Since the decomposition in a basis is unique, (4) provides a bijection $\rho \mapsto b$ between $\mathcal{E}$ and the unit ball in $\mathbb{R}^3$. Finally, $\rho$ is pure iff its spectrum is $\{ 0, 1 \}$, which is equivalent to $\| b \| = 1$, corresponding to a point on the sphere.