Problem 1. (i) We compute
\[ \left\| \sum_{n=0}^{N} \left( \frac{A}{\lambda} \right)^n \right\| \leq \sum_{n=0}^{N} \left( \frac{\|A\|}{|\lambda|} \right)^n \]
which is convergent iff \( \|A\| < |\lambda| \). In that case,
\[ (\lambda I - A) \frac{1}{\lambda} \sum_{n=0}^{N} \left( \frac{A}{\lambda} \right)^n = I - \left( \frac{A}{\lambda} \right)^{N+1} \rightarrow I \quad (N \rightarrow \infty), \]
in norm, proving that \( \frac{1}{\lambda} \sum_{n=0}^{\infty} \left( \frac{A}{\lambda} \right)^n \) is the right inverse of \( \lambda I - A \). The same calculation shows that it is also its left inverse.

Remark. In this context, the series is called the Neumann serie.

(ii) Let now \( A = A^* \) and let \( \lambda = x + iy \in \sigma(A) \), with \( x, y \in \mathbb{R} \) and \( x^2 + y^2 \leq \|A\|^2 \) by (i). We claim that \( y = 0 \). Let \( t \in \mathbb{R} \). By the C*-property and \( A = A^* \),
\[ \|(A + itI)(A - itI)\| = \|A^2 + t^2I\| \leq \|A\|^2 + t^2. \]
Since \( \lambda + it \in \sigma(A + itI) \), the inequality above and (i) imply that
\[ |x + iy + t|^2 \leq \|A\|^2 + t^2, \]
namely
\[ 2yt \leq \|A\|^2 - x^2 - y^2. \]
Since this holds for all \( t \in \mathbb{R} \), we must have that \( y = 0 \).

(iii) Let \( A \in \mathcal{I} \) and let \( B \) be such that \( \|B - A\| < \frac{1}{\|A - 1\|} \). By (i), it follows that \( I - A^{-1}(A - B) \) is invertible, so that \( B \) is invertible and hence \( \mathcal{I} \) is open. Furthermore, again by (i)
\[ \|A^{-1} - B^{-1}\| = \left\| A^{-1} - \sum_{n=0}^{\infty} (A^{-1}(A - B))^n A^{-1} \right\| \]
\[ \leq \|A^{-1}\| \sum_{n=1}^{\infty} \|A^{-1}(A - B)\|^n \]
\[ = \frac{\|A^{-1}\|^2 \|A - B\|}{1 - \|A^{-1}\| \|A - B\|} < 2\|A^{-1}\|^2 \|A - B\| \]
if \( \|A - B\| < \frac{1}{2\|A - 1\|} \), proving continuity.

Problem 2. (i) Since the quadratic form
\[ \mathbb{C} \ni \lambda \mapsto \omega((A + \lambda B)^* (A + \lambda B)) = \omega(A^* A) + \overline{\lambda} \omega(B^* A) + \lambda \omega(A^* B) + |\lambda|^2 \omega(B^* B) \]
is non-negative, it is in particular real so that \( \omega(B^*A) - \omega(A^*B) = 0 \), which is the second equality. Choosing \( \lambda \in \mathbb{R} \), non-negativity implies that the discriminant is always non-positive,

\[
4\text{Re}(\omega(A^*B))^2 - 4\omega(B^*B)\omega(A^*A) \leq 0,
\]
which yields the Cauchy-Schwarz inequality by noting that \(|\text{Re}(z)| \leq |z|\). To prove the last one, we first apply Cauchy-Schwarz to obtain

\[
|\omega(A^*BA)|^2 \leq \omega(A^*A)\omega(A^*B^*BA).
\]  

We then note that the inequality \( A^*B^*BA \leq \|B\|^2A^*A \) together with the positivity and linearity of the state implies that

\[
\omega(A^*B^*BA) - \|B\|^2\omega(A^*A) = \omega(A^*B^*BA - \|B\|^2A^*A) \leq 0
\]
which yields the claim when used in (1). Note that \( \sigma(B^*B - \|B\|^2I) = \sigma(B^*B) - \|B^*B\| \subset [-2\|B^*B\|, 0] \) implies that \( B^*B - \|B\|^2I \leq 0 \). In particular, there is \( D \) such that \( B^*B - \|B\|^2I = -D^*D \). But then \( A^*B^*BA - \|B\|^2A^*A = -A^*D^*DA = -(DA)^*(DA) \leq 0 \).

(ii) It suffices to apply the inequality

\[
\omega(X^*X)\omega(Y^*Y) \geq |\omega(X^*Y)|^2 \geq |\text{Im}(\omega(X^*Y))|^2 = \frac{1}{4}|\omega(X^*Y - Y^*X)|^2
\]
to the self-adjoint \( X = A - \omega(A), Y = B - \omega(B) \) since \([X, Y] = [A, B]\).

Remark. If \( \mathcal{A} \) is an Abelian algebra, then the right hand side of the inequality vanishes for all \( A, B \in \mathcal{A} \). This is the case for classical systems.

**Problem 3.** (i) Immediate by the definition of a Schwartz function.

(ii) We first note that \( Q, P \) are symmetric operators on \( \mathcal{S} \), namely

\[
\langle f, Qf \rangle = \langle Qf, f \rangle, \quad \langle f, Pf \rangle = \langle Pf, f \rangle
\]
for all \( f \in \mathcal{S} \), where the second equality holds by integration by parts and the fast decay of Schwartz functions at infinity. (i) and the Cauchy-Schwarz inequality for the \( L^2 \)-scalar product \( |\langle f, g \rangle| \leq ||f|| ||g|| \) imply

\[
|\omega_f(Q^*P)|^2 = |\langle Qf, Pf \rangle|^2 \leq ||Qf||^2 ||Pf||^2 = \omega_f(Q^*Q)\omega_f(P^*P),
\]

namely the eponymous inequality for the state \( \omega_f \). In particular, the Heisenberg inequality holds for \( f \in \mathcal{S} \) with

\[
\omega_f([Q, P]) = \langle f, QPf \rangle - \langle f, PQf \rangle = -i \int_{\mathbb{R}} \overline{f(x)}(xf'(x) - \frac{d}{dx}(xf(x)))\,dx = i\|f\|^2 = i.
\]

(note that all operations are well-defined by (i) and \( f \in \mathcal{S} \)). We conclude as in Problem 2 that

\[
\text{Var}_{\omega_f}(Q)\text{Var}_{\omega_f}(P) \geq \frac{1}{4},
\]
which is the celebrated Heisenberg’s Uncertainty Principle.

**Remark.** This setting is that of a single particle moving on the line. The measure \( d\rho(x) = |f(x)|^2\,dx \)
has the physical interpretation of the probability density that the particle is around \( x \). Its first moment, which is \( \langle f, Qf \rangle \), is the expectation value of the position of the particle. After Fourier transformation, \( d\mu(k) = |\hat{f}(k)|^2dk \) is the probability density for the momentum of the particle. Hence, Heisenberg’s inequality is a lower bound on the product of the variances of position and momentum of the particle, which is independent of the wavefunction \( f \).

(iii) It suffices to consider the case \( \omega_f(Q) = \omega_f(P) = 0 \). Indeed, if \( p = \omega_f(P), q = \omega_f(Q) \), these expectation values vanish for the function \( x \mapsto e^{-ipx} f(x+q) \). Equality holds in Heisenberg’s bound iff the following equalities hold:

\[
|\text{Im} \langle Qf, Pf \rangle| = |\langle Qf, Pf \rangle|, \quad |\langle Qf, Pf \rangle| = \|Qf\|\|Pf\|,
\]

see (2,3), namely

\[
\langle Qf, Pf \rangle \in i\mathbb{R}, \quad Qf = \mu Pf \text{ for } \mu \in \mathbb{C}.
\]

By the second condition, \( f \) solves the differential equation

\[
f'(x) = \frac{i}{\mu} xf(x)
\]

namely \( f(x) = ce^{\frac{ix^2}{2\mu}} \). Now

\[
\langle Qf, Pf \rangle = \frac{|c|^2}{\mu} \int_{\mathbb{R}} xe^{i\frac{x^2}{\mu}} dx \in i\mathbb{R}
\]

iff \( -\xi = \frac{1}{\mu} < 0 \), namely \( f_0(x) = ce^{-\xi x^2} \). By the initial remark, the general solution is given by

\[
f_{p,q}(x) = e^{-ipx} f_0(x+q) \quad (p, q \in \mathbb{R}).
\]

**Problem 4.** (i) Since \( \mathcal{H} \) is finite dimensional, so is \( \mathcal{B}(\mathcal{H}) \) and so is the set of linear functionals over \( \mathcal{B}(\mathcal{H}) \). Hence a state is completely characterized by its action on the canonical basis \( \{e^{ij} : 1 \leq i, j \leq \dim(\mathcal{H})\} \) and the claim follows from the equality

\[
\omega(e^{ij}) = \rho_{j,i}
\]

together with the fact that \( \omega(1) = \sum_{i=1}^{\dim(\mathcal{H})} \omega(e^{ii}) = \text{Tr}(\rho) \), which shows the equivalence of the trace condition for \( \rho \) and the normalization of the state \( \omega \).

(ii) We consider the corresponding density matrix \( \rho \) and its spectral decomposition

\[
\rho = \sum_{i=1}^{\dim(\mathcal{H})} \rho_i P_{\psi_i}
\]

where \( P_\phi \) is the one-dimensional projection onto the span of \( \phi \), \( \{\rho_i : 1 \leq i \leq \dim(\mathcal{H})\} \) are the eigenvalues and \( \{\psi_i : 1 \leq i \leq \dim(\mathcal{H})\} \) is a corresponding orthonormal set. Since \( P_{\psi_i} \) is itself a density matrix, and \( 1 = \text{Tr}(\rho) = \sum_i \rho_i \), the spectral decomposition provides a convex decomposition of the state \( \omega \). Hence, if \( \omega \) is pure, then \( \rho_{i_0} = 1, \rho_i = 0 (i \neq i_0) \) and \( \rho = P_{\psi_{i_0}} \) indeed. Reciprocally, if \( \rho = P_\psi \), all but one eigenvalues of \( \rho \) vanish and the claim follows by uniqueness of the spectral decomposition.
(iii) We express a density matrix in $\mathcal{M}_2(\mathbb{C})$ in the basis given by the identity and the three Pauli matrices:

$$\rho = \frac{1}{2}(aI + \sum_{j=1}^{3} b_j\sigma^j).$$

(4)

Since the latter are traceless, we have that $a = 1$ by the normalization condition. $\rho$ is self-adjoint iff $b_j \in \mathbb{R}$. The eigenvalues of $b \cdot \sigma = \sum_{j=1}^{3} b_j\sigma^j$ are given by $\pm\|b\|$ so that those of $\rho$ are given by $\frac{1}{2}(1 \pm \|b\|)$. Hence $0 \leq \rho \leq I$ if and only if $\|b\| \leq 1$. Since the decomposition in a basis is unique, (4) provides a bijection $\rho \mapsto b$ between $\mathcal{E}$ and the unit ball in $\mathbb{R}^3$. Finally, $\rho$ is pure iff its spectrum is $\{0, 1\}$, which is equivalent to $\|b\| = 1$, corresponding to a point on the sphere.