You are strongly encouraged to work on all three problems of this set. However, I will grade only two problems of your choice. Please indicate clearly on your solution sheet which problems you want to be considered.

On this sheet, the Hilbert space $\mathcal{H}$ is finite dimensional.

**Problem 1.** Let $P, Q$ be two projections on $\mathcal{H}$. Let

$$R = (P - Q)^2, \quad \tilde{U} = QP + (1 - Q)(1 - P), \quad \tilde{V} = PQ + (1 - P)(1 - Q),$$

and

$$U = \tilde{U}(1 - R)^{-1/2}, \quad V = \tilde{V}(1 - R)^{-1/2},$$

whenever these operators are well-defined.

(i) Prove that $R$ commutes with both $P$ and $Q$.

(ii) Prove that $[\tilde{U}, \tilde{V}] = 0$ and that $\tilde{U}\tilde{V} = 1 - R$.

From now on, assume that $P, Q$ are orthogonal projections and that $\|R\| < 1$.

(iii) Prove that $U, V$ are unitary with $V^* = U$.

(iv) Prove that

$$Q = UPU^*.$$

(v) Let $\mathcal{D}$ be a connected region of $\mathbb{C}$, and let $\mathcal{D} \ni z \mapsto P(z)$ be a continuous function such that $P(z) = P(z)^2 = P(z)^*$ for all $z \in \mathcal{D}$. Prove that $\text{Rank}(P(z))$ is constant.

**Problem 2.** We say that $M(z)$ is an analytic function (in a neighbourhood of $z = 0$) if the Taylor series $M(z) = \sum_{j=0}^{\infty} z^n M_n$ is norm-convergent for all $|z| < \rho$ for some $\rho > 0$. We denote

$$M = M(0), \quad N(z) = M(z) - M.$$

Let

$$R(\zeta, z) = (M(z) - \zeta)^{-1}.$$

(i) Assume that $\zeta_0$ is not in the spectrum of $M$. Prove that if $|\zeta - \zeta_0|$ is sufficiently small, $R(\zeta, z) = R(\zeta_0)(1 - (\zeta - \zeta_0 - N(z))R(\zeta_0))^{-1}$.

(ii) Prove that $z \mapsto R(\zeta, z)$ is analytic, namely

$$R(\zeta, z) = R(\zeta) + \sum_{n=1}^{\infty} z^n R_n(\zeta)$$

for $|z|$ sufficiently small, and give a formula for the coefficients $R_n(\zeta)$.

(iii) Let $\lambda$ be an eigenvalue of $M$, and let $\Gamma$ be a positively oriented circle in $\mathbb{C}$ that encloses $\lambda$ and
no other eigenvalue. Prove that if $|z|$ is sufficiently small, then $M(z)$ has no eigenvalue on $\Gamma$.
(iv) Prove that if $|z|$ is sufficiently small the projection

$$P(z) = -\frac{1}{2\pi i} \int_\Gamma R(\zeta, z) d\zeta$$

is well-defined and analytic.
(v) Conclude that if $|z|$ is sufficiently small, then $\text{Rank}(P(z))$ is constant.

**Problem 3.** The Hamiltonian for a spin-$\frac{1}{2}$ in a magnetic field $-B \in \mathbb{R}^3$ is given by

$$H_B = B \cdot \sigma,$$

where $\sigma$ is the vector of Pauli matrices.

(i) Compute the eigenvalues $\lambda_B^\pm$ (where $\lambda_B^- \leq \lambda_B^+$) and normalized eigenvector $\Omega_B^-$ of $H_B$. Adjust the phase so that the first component is real.
(ii) Assume that $B \neq 0$. Compute Berry’s connection

$$A = \langle \Omega_B^-, d\Omega_B^- \rangle,$$

in the coordinates $(B_1, B_2, B_3)$, where $d$ is the exterior derivative:

$$df = \sum_{j=1}^3 \frac{\partial f}{\partial B_j} dB_j.$$

Let now $\|B\| = \hbar$ be fixed. The parameter manifold is $\mathcal{M} = S^2$.

(iii) Express $\Omega_B^-$ in polar coordinates. Characterize the domain $\mathcal{C} \subset S^2$ of this representation.
(iv) Consider now $\tilde{\Omega}_B^- = e^{-i\phi} \Omega_B^-$ and characterize the domain $\tilde{\mathcal{C}} \subset S^2$ of this representation. Compute the corresponding $A, \tilde{A}$ in polar coordinates and prove that

$$dA = d\tilde{A}, \quad \text{on } \mathcal{C} \cap \tilde{\mathcal{C}}.$$

(v) Find a function $t : S^1 \to U(1)$ such that

$$A = t^{-1} \tilde{A} t + t^{-1} dt.$$

(vi) Show by an explicit calculation that the total flux is quantized:

$$\frac{1}{2\pi i} \int_{S^2} dA = -1.$$