You are strongly encouraged to work on all four problems of this set. However, I will grade only two problems of your choice. Please indicate clearly on your solution sheet which problems you want to be considered.

Unless otherwise stated, all $C^*$-algebras have a unit.

**Problem 1.** Let $A$ be a $C^*$-algebra and let $\omega$ be a state over $A$.

(i) Let $N = \{ A \in A : \omega(A^*A) = 0 \}$ prove that $A \in A, N \in N$ implies that $AN \in N$.

(ii) For any $A \in A$, denote $\psi_A = \{ A \in A : \exists N \in N : A = A + N \}$ and let $h$ be the set of such equivalence classes. Prove that

$$\langle \psi_A, \psi_B \rangle = \omega(A^*B)$$

is independent of the representatives $A, B$ and defines an inner product on $h$.

(iii) Prove that the linear map $\pi : A \to B(h)$ defined by

$$\pi(A)\psi_B = \psi_{AB}$$

is bounded, and a *-morphism, namely

$$\pi(A^*) = \pi(A)^*, \quad \pi(AB) = \pi(A)\pi(B).$$

(iv) Denote $\Omega = \psi_I \in h$. Prove that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle. \quad (1)$$

**Remark.** $h$ can be completed to a Hilbert space $H$ and $\pi$ extended by boundedness to all of $H$. The triple $(H, \pi, \Omega)$ is called the GNS representation of $A$ associated with $\omega$. In particular, the set $\{ \pi(A)\Omega : A \in A \}$ is dense in $H$. This is a useful property for Problem 2.

**Problem 2.** Same assumptions as in Problem 1.

(i) Prove that the GNS representation is unique up to unitary equivalence. More precisely, given any two representations $(H_i, \pi_i, \Omega_i), i = 1, 2$, satisfying (1) for the same state $\omega$, construct a unitary intertwiner $U : H_1 \to H_2$:

$$U\pi_1(A) = \pi_2(A)U, \quad U\Omega_1 = \Omega_2.$$

(ii) Let $\alpha$ be a *-automorphism of $A$ and $\omega$ be an $\alpha$-invariant state:

$$\omega \circ \alpha = \omega.$$

Prove that there is a unique unitary operator $U$ on the GNS Hilbert space $H_{\omega}$ such that $\pi_{\omega}(A)U = U\pi_{\alpha}(\alpha(A))$ for $A \in A$ and $U\Omega_{\omega} = \Omega_{\omega}$.

**Hint:** Use (i).
Problem 3. Consider the C*-algebra $A = B(H)$ of bounded linear operators on a Hilbert space $H$. Let $\rho = \rho^*$ be a strictly positive density matrix on $H$, namely $\rho^* \leq I$, $\text{Tr} (\rho) = 1$. Let 
$$\omega(A) = \text{Tr} (\rho A)$$
be the corresponding state. Consider the triple $(H_\rho, \pi_\rho, \Omega_\rho)$ defined as follows:

- $H_\rho$ is the set of Hilbert-Schmidt operators (namely those $T \in B(H)$ such that $\text{Tr}(T^*T) < \infty$), with the scalar product $\langle T, S \rangle_{H_\rho} := \text{Tr}(T^*S)$.
- For each $A \in A$, let $\pi_\rho(A) : H_\rho \rightarrow H_\rho$ be defined by $\pi_\rho(A)T := AT$ for $T \in H_\rho$.
- $\Omega_\rho := \rho^{1/2}$.

(i) Prove that $\Omega_\rho$ is a unit vector in $H_\rho$.
(ii) Prove that $\pi_\rho$ is a faithful representation of $A$ into $B(H_\rho)$, namely that $\text{Ker} (\pi_\rho) = \{0\}$.

**Hint:** In any Hilbert space, $\|v\| = \sup_{\|u\|=1} |\langle u, v \rangle|$. 
(iii) Prove that $\rho_\rho$ is a cyclic vector for the representation $\pi_\rho$, i.e. that the set $\{\pi_\rho(A)\Omega_\rho : A \in A\}$ is dense in $H_\rho$.
(iv) Prove that $\omega(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle$ for all $A \in A$.

Problem 4. Let $A$ be the CAR algebra over the Hilbert space $H$.
(i) Use the CAR to prove that $\|b(f)\| = \|b^*(f)\| = \|f\|_H$.
(ii) Use the CAR to compute the spectrum of $b(f), b^*(f)$ and $b^*(f)b(f)$.
(iii) Let now $H = l^2(\Lambda)$, where $\Lambda$ is a finite set and let $N_\Lambda = \sum_{x \in \Lambda} b_x^*b_x$ where $b_x = b(\delta_x)$ acting on Fock space. Prove that $\sigma(N) = \{0, 1, \ldots |\Lambda|\}$.
(vi) Prove that for any $\theta \in \mathbb{R}$, 
$$e^{i\theta N}b(f)e^{-i\theta N} = b(e^{i\theta} f).$$

**Hint.** Uniqueness for the solution of an ODE. Note that the map $f \mapsto b(f)$ is antilinear.
(v) Let $A_X^+$ be the even subalgebra of $A$ and let $A \in A_X^+, B \in A_Y$. Prove that $[A, B] = 0$ whenever $X \cap Y = \emptyset$.

**Hint.** Reduce to monomials in creation / annihilation operators.