Homework set 1 – due January 22, 2019

You are strongly encouraged to work on all four problems of this set. However, I will grade only two problems of your choice. Please indicate clearly on your solution sheet which problems you want to be considered.

Unless otherwise stated, all C*-algebras have a unit.

Problem 1. Let \( A \) be a C*-algebra and \( A \in A \).

(i) Prove that if \( \lambda \in \mathbb{C} \) is such that \( |\lambda| > \|A\| \), then the sum

\[
\sum_{n=0}^{\infty} \left( \frac{A}{\lambda} \right)^n
\]

is convergent and that \( \lambda I - A \) is invertible (with inverse in \( A \)).

(ii) The spectrum \( \sigma(A) \) of \( A \in A \) is the set \( \lambda \in \mathbb{C} \) such that \( \lambda I - A \) is not invertible. Prove that if \( A = A^* \) then \( \sigma(A) \subset [-\|A\|, \|A\|] \).

(iii) Let \( I \subset A \) be the set of invertible elements of \( A \). Prove that \( I \) is an open set and that the map \( A \mapsto A^{-1} \) is continuous on \( I \).

Hint. \( B = A(I - A^{-1}(A - B)) \) whenever \( A \in I \).

Problem 2. Let \( A \) be a C*-algebra and \( \omega : A \to \mathbb{C} \) be a state.

(i) Prove the Cauchy-Schwarz inequality: For any \( A, B \in A \),

\[
|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B),
\]

as well as

\[
\omega(A^*B) = \overline{\omega(B^*A)}
\]

and

\[
|\omega(A^*BA)| \leq \|B\|\omega(A^*A).
\]

Hint. You may need the following facts about positive elements of a C*-algebra. \( A \in A \) is called positive if \( A = A^* \) and \( \sigma(A) \subset [0, \infty) \). In that case, the integral

\[
\pi \int_{0}^{\infty} \sqrt{\lambda} (\lambda^{-1} - (\lambda I + A)^{-1}) \, d\lambda
\]

is convergent and defines a self-adjoint operator \( S \) such that \( S^2 = S^*S = A \). Reciprocally, any element of the form \( A^*A \) is a positive element of the algebra.

(ii) Let \( \text{Var}_\omega(A) = \omega((A - \omega(A)I)^*(A - \omega(A)I)) \). Prove the Heisenberg inequality

\[
\text{Var}_\omega(A)\text{Var}_\omega(B) \geq \frac{1}{4} |\omega([A, B])|^2,
\]

for any self-adjoint elements \( A = A^* \in A, B = B^* \in A \).
Problem 3. Let $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C})$ be the space of square-integrable functions with scalar product and norm
\[
\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx, \quad \|f\| = \left( \int_{\mathbb{R}} |f(x)|^2dx \right)^{\frac{1}{2}}.
\]
Let $\mathcal{S}$ be the space of Schwartz functions:
\[
\mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} \left| x^n \frac{d^m}{dx^m} f(x) \right| < \infty \text{ for all } n, m \in \mathbb{N} \right\} \subset \mathcal{H}.
\]
Finally let $Q, P$ be defined by
\[
(Qf)(x) = xf(x), \quad (Pf)(x) = -if'(x).
\]

(i) Prove that $QS \subset \mathcal{S}$ and $PS \subset \mathcal{S}$.

(ii) For any $f \in \mathcal{S}$ with $\|f\| = 1$, let $\omega_f(\cdot) = \langle f, (\cdot)f \rangle$. Prove that
\[
\text{Var}_{\omega_f}(Q)\text{Var}_{\omega_f}(P) \geq \frac{1}{4}.
\]

(iii) Prove that equality holds above if and only if
\[
f(x) = ce^{-ipx}e^{-\xi(x+q)^2},
\]
where $c \in \mathbb{C}, \xi > 0$ and $p, q \in \mathbb{R}$.

Hint. Consider first the case $\omega_f(Q) = \omega_f(P) = 0$.

Problem 4. Let $\mathcal{H}$ be a finite-dimensional Hilbert space.

(i) Prove that there is a one-to-one correspondence between states (positive, linear functionals over $B(\mathcal{H})$ such that $\omega(I) = 1$) and density matrices (positive elements of $B(\mathcal{H})$ with unit trace).

(ii) A state $\omega$ is said to be pure if it cannot be decomposed as a convex combination of other states, namely
\[
\omega = \lambda \omega_1 + (1 - \lambda)\omega_2, \lambda \in (0, 1) \implies \omega = \omega_1 = \omega_2.
\]

Prove that $\omega$ is pure if and only if the corresponding density matrix is a one-dimensional projection.

Hint. Spectral theorem.

(iii) Let now $\mathcal{H} = \mathbb{C}^2$ and let $\mathcal{E}_2$ be the set of states over $\mathcal{H}$. Prove that there is a bijection $b : \mathcal{E}_2 \mapsto \{ x \in \mathbb{R}^3 : \|x\| \leq 1 \}$ such that $\omega \in \mathcal{E}_2$ is pure if and only if $\|b(\omega)\| = 1$.

Hint. Pauli matrices.