Problem 1. Let $\mathcal{A}$ be a C*-algebra and $A \in \mathcal{A}$.
(i) Prove that if $\lambda \in \mathbb{C}$ is such that $|\lambda| > \|A\|$, then the sum
\[ \sum_{n=0}^{\infty} \left( \frac{A}{\lambda} \right)^n \]
is convergent and that $\lambda \mathbb{1} - A$ is invertible (with inverse in $\mathcal{A}$).
(ii) The spectrum $\sigma(A)$ of $A \in \mathcal{A}$ is the set $\lambda \in \mathbb{C}$ such that $\lambda \mathbb{1} - A$ is not invertible. Prove that if $A = A^*$ then $\sigma(A) \subset [-\|A\|, \|A\|]$.
(iii) Let $I \subset \mathcal{A}$ be the set of invertible elements of $\mathcal{A}$. Prove that $I$ is an open set and that the map $A \mapsto A^{-1}$ is continuous on $I$.

Hint. $B = A(\mathbb{1} - A^{-1}(A - B))$ whenever $A \in I$.

Problem 2. Let $\mathcal{A}$ be a C*-algebra and $\omega : \mathcal{A} \to \mathbb{C}$ be a state.
(i) Prove the Cauchy-Schwarz inequality: For any $A, B \in \mathcal{A}$,
\[ |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B), \]
as well as
\[ \omega(A^*B) = \overline{\omega(B^*A)} \]
and
\[ |\omega(A^*BA)| \leq \|B\|\omega(A^*A). \]
(ii) Let $\text{Var}_\omega(A) = \omega((A - \omega(A)\mathbb{1})^*(A - \omega(A)\mathbb{1}))$. Prove the Heisenberg inequality
\[ \text{Var}_\omega(A)\text{Var}_\omega(B) \geq \frac{1}{4} |\omega([A, B])|^2, \quad [A, B] = AB - BA \]
for any self-adjoint elements $A = A^* \in \mathcal{A}, B = B^* \in \mathcal{A}$. 
Problem 3. Let $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C})$ be the space of square-integrable functions with scalar product and norm
\[
\langle f, g \rangle = \int_\mathbb{R} f(x)g(x)dx, \quad \| f \| = \left( \int_\mathbb{R} |f(x)|^2dx \right)^{\frac{1}{2}}.
\]
Let $\mathcal{S}$ be the space of Schwartz functions:
\[
\mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} \left| x^n \frac{d^m}{dx^m} f(x) \right| < \infty \text{ for all } n, m \in \mathbb{N} \right\} \subset \mathcal{H}.
\]
Finally let $Q, P$ be defined by
\[
(Qf)(x) = xf(x), \quad (Pf)(x) = -if'(x).
\]
(i) Prove that $Q\mathcal{S} \subset \mathcal{S}$ and $P\mathcal{S} \subset \mathcal{S}$.
(ii) For any $f \in \mathcal{S}$ with $\| f \| = 1$, let $\omega_f(\cdot) = \langle f, (\cdot)f \rangle$. Prove that
\[
\text{Var}_{\omega_f}(Q)\text{Var}_{\omega_f}(P) \geq \frac{1}{4}.
\]
(iii) Prove that equality holds above if and only if
\[
f(x) = ce^{-ipx}e^{-\xi(x+q)^2},
\]
where $c \in \mathbb{C}, \xi > 0$ and $p, q \in \mathbb{R}$.
*Hint.* Consider first the case $\omega_f(Q) = \omega_f(P) = 0$.

Problem 4. Let $\mathcal{H}$ be a finite-dimensional Hilbert space.
(i) Prove that there is a one-to-one correspondence between states (positive, linear functionals over $B(\mathcal{H})$ such that $\omega(\mathbb{I}) = 1$) and density matrices (positive elements of $B(\mathcal{H})$ with unit trace).
(ii) A state $\omega$ is said to be pure if it cannot be decomposed as a convex combination of other states, namely
\[
\omega = \lambda \omega_1 + (1 - \lambda) \omega_2, \ \lambda \in (0, 1) \implies \omega = \omega_1 = \omega_2.
\]
Prove that $\omega$ is pure if and only if the corresponding density matrix is a one-dimensional projection.
*Hint.* Spectral theorem.
(iii) Let now $\mathcal{H} = \mathbb{C}^2$ and let $\mathcal{E}_2$ be the set of states over $\mathcal{H}$. Prove that there is a bijection $b : \mathcal{E}_2 \mapsto \{ x \in \mathbb{R}^3 : \| x \| \leq 1 \}$ such that $\omega \in \mathcal{E}_2$ is pure if and only if $\| b(\omega) \| = 1$.
*Hint.* Pauli matrices.